Some Physical And Mathematical Aspects Of Non-commutative Geometry

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By

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To my beloved brother Partha Sarathi Chakraborty,
who has always instilled in me a thirst for knowledge.

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Abstract

The Standard Model (SM) of particle physics seems to be inadequate at high energy scale presumably in the vicinity of Planck energy scale $E_p = \sqrt{\frac{\hbar c^5}{G}} \sim 10^{19} GeV$, because it ignores quantum nature of gravity and at Planck energy scale, a combined effects of both quantum and gravitational effects are expected to become more pronounced. Scientists are employing diverse strategies to study Quantum Gravity and aiming to unify General Relativity and Quantum Mechanics. While all these theories are still in their nascent stages of development and haven't achieved any major breakthroughs, they all point towards a common feature: the quantum nature of space-time at the Planck scale. In certain models of quantum space-time in non-commutative geometrical framework, the space-time coordinates become operator valued satisfying a non-vanishing commutator algebra involving certain deformation parameter(s). These parameter(s) are believed to correspond to a minimal length scale, presumably the Planck length scale.

Although no effort will be made in the current thesis to directly address any of the problems on Quantum Gravity, this thesis rather adds to a thorough investigation of different physical and mathematical aspects of non-commutative (NC) space-time.

The aim of this thesis is two-fold. In part-A, we have taken prototypes of NC space-time and tried to build dynamical models on such a background and discuss its effect on the respective systems for both relativistic and non-relativistic cases.

- First we discuss the formulation of non-relativistic quantum mechanics on NC space-time of Moyal type in 1+1 dimension, using Hilbert Schmidt operators, where 'time' is also taken to be an operator. We then study its impact on a time-dependent quantum system to demonstrate, in particular, the possibility of emergence of geometric phase, which depends on the NC parameter, after the system has undergone a periodic and adiabatic evolution.

- Next using the deformed Poincare symmetry of the κ-Minkowski space-time (a Lie-algebraic type of non-commutative Lorentzian space-time), we construct a Lagrangian for a spin-less relativistic massive particle, and explored a conceivable regime of a future theory of quantum gravity in which the momentum space becomes curved. Finally, an attempt is made to illustrate that the corresponding deformed geodesic distance in the momentum space impacts the single particle dispersion relation by deforming it, and it is further observed that the original 'bare' mass of the particle gets "renormalized" as a result of noncommutativity.

In part-B, we primarily discuss the spectral triple approach of non-commutative geometry (NCG).

- There has been a recent upsurge of activities where attempts are being made to construct spectral triple for a manifold with Lorentzian (i.e. pseudo-Riemannian) signature. We built the spectral triple for 1+1 dimensional Lorentzian Moyal space-time using the pseudo-Riemannian spectral triple formulation provided recently by N Franco *et.al.* and then computed the spectral distance between a pair of pure states separated by a time-like interval in such space-time.

- Finally, we construct a real structure for the fuzzy sphere S^2_* in its lowest spin-1/2 representation by enlarging the symmetry group from SO(3) to O(3). Here, the NC coordinates satisfy $\mathfrak{su}(2)$ algebra, so that we have a real and even spectral triple on S^2_*. We show that the first order condition on the Dirac operator is violated, which should pave the way to construct an $\mathfrak{su}(2)$ gauge theory as a toy model and is expected to help us to study physics beyond the standard model of particle physics.

সারাংশ

উচ্চ শক্তির পদার্থবিদ্যার অনুমান অনুযায়ী প্ল্যাঙ্ক শক্তি তথা $১০^{১৯}$ গিগা ইলেক্ট্রন ভোল্ট বা তার বেশি শক্তিস্তরে, কোয়ান্টাম এবং মহাকর্ষীয় বল উভয়ের প্রভাব তাৎপর্যপূর্ণ হইয়া ওঠে। পদার্থবিদ্যায় অতি পারমাণবিক কণার ক্ষেত্রে প্রযোজ্য স্ট্যান্ডার্ড মডেলে মহাকর্ষ বলের কোয়ান্টাম চরিত্রকে উপেক্ষা করা হয়। ফলত উচ্চ শক্তির মাত্রায় এই মডেল অপর্যাপ্ত হইয়া পড়ে। বহুকাল যাবৎ সাধারণ আপেক্ষিকতাবাদ এবং কোয়ান্টাম বলবিদ্যার ঐক্য সাধনের উদ্দেশ্য লইয়া বিভিন্ন বিজ্ঞানী বিভিন্ন আঙ্গিকে কোয়ান্টাম মহাকর্ষীয় তত্ত্ব খাড়া করিবার চেষ্টা করিতেছেন। যদিও এই তত্ত্বগুলির মধ্যে কোনটিই তাহাদের সমাধানের শীর্ষে পৌঁছায় নাই, তথাপি উচ্চ শক্তির মাত্রায় গড়িয়া ওঠা সকল তত্ত্বেই প্রায় একটি মিল রহিয়াছে - সেইটি হইল প্লাঙ্ক স্তরে দেশ-কালের কোয়ান্টাম চরিত্র। কিছু তত্ত্বে ধরিয়া লইয়া হয় যে, উচ্চ শক্তি মাত্রায় স্থানাঙ্ক জ্যামিতি আমাদের সাধারন ধারনার বাইরে বিকৃত রূপ ধারন করে এবং দেশ-কালের স্থানাঙ্ক গুলি অপারেটরের স্তরে উন্নীত হইয়া নিজেদের মধ্যে নন্-কমিউটেটিভ বীজগানিতিক সম্পর্ক (non-commutative algebra) তৈরি করে। উল্লেখ্য, দেশ-কালের এই বিকৃতির স্থিতিমাপ (parameter) একটি ন্যুনতম দৈর্ঘ্য তথা প্ল্যাঙ্ক-দৈর্ঘ্যের সমান বলিয়া ধরিয়া লওয়া হয়।

বর্তমান থিসিস এ যদিও কোয়ান্টাম মাধ্যাকর্ষণ বিদ্যা সংক্রান্ত কোনো সমস্যা সরাসরি সমাধানের জন্য প্রচেষ্টা করা হয় নাই, তথাপি এই থিসিসটিতে নন্-কমিউটেটিভ দেশ-কালের বিভিন্ন তাত্ত্বিক এবং গাণিতিক দিকগুলির একটি পুঙ্খানুপুঙ্খ তদন্ত করা হইয়াছে। এই থিসিসের উদ্দেশ্য দুই ভাগে বিভক্ত করা যায়।

প্রথম ভাগে, আমরা নন্-কমিউটেটিভ দেশ-কালের কিছু নমুনাকে প্রাথমিক হিসাবে ধরিয়া লইয়া এই ধরনের পটভূমিতে আপেক্ষিক (relativistic) ও অনাপেক্ষিক (non-relativistic) উভয় ক্ষেত্রেই সংশ্লিষ্ট সিস্টেমে গতিবিদ্যায় তাহার প্রভাব লইয়া আলোচনা করিয়াছি।

- প্রথমে আমরা হিলবার্ট-স্মিড্ (Hilbert-Schmidt) অপারেটর ব্যবহার করিয়া ১+১ মাত্রায় ময়াল (Moyal) প্রকৃতির নন্-কমিউটেটিভ তলে অনাপেক্ষিক কোয়ান্টাম বলবিদ্যা লইয়া আলোচনা করিয়াছি, যেখানে 'সময়'কেও অপারেটর হিসেবে ধরিয়া লইয়া হইয়াছে এবং সময়-নির্ভর কোয়ান্টাম বলবিদ্যার উপর নন্-কমিউটেটিভ স্থিতিমাপ এর প্রভাব অধ্যয়ন করা হইয়াছে। এখানে রুদ্ধতাপী প্রক্রিয়ায় সিস্টেমের বিবর্তনের পরে জ্যামিতিক দশা তথা প্রচলিত ভাষায় বেরি ফেজের উৎপত্তি দেখানো হইয়াছে।

- পরবর্তীতে, কাপ্পা-মিন্কওস্কি (kappa-Minkowski) (যেটি একটি লি-বীজগাণিতিক ধরনের নন-কমিউটেটিভ লরেন্টজিয় জ্যামিতি) দেশ-কালের পোঁয়াকারে (Poincare) প্রতিসাম্য ব্যবহার করিয়া, আমরা একটি কৌণিক ভরবেগ বিহীন (spin-less) অথচ ভর যুক্ত আপেক্ষিক কণার জন্য ল্যাগ্রাঞ্জিয়ান তৈরি করিয়াছি। ইহার দ্বারা একটি গ্রহণযোগ্য কোয়ান্টাম মহাকর্ষের তত্ত্ব তৈরীর চেষ্টা করা হইয়াছে যেখানে ভরবেগের উপাংশগুলি একটি বক্র জ্যামিতিক গঠনকে (curved momentum space) নির্দেশ করে। ইহার পরে এই বক্র জ্যামিতিতে বিকৃত জিওডেসিক দূরত্বকে একটি বিকৃত বিচ্ছুরণের সম্পর্ক (deformed dispersion relation) হিসেবে ধরিয়া লওয়া হয় এবং পরিশেষে, এই বিকৃতিকে নন্-কমিউটেটিভ স্থিতিমাপের সঙ্গে সম্পর্কযুক্ত করা হয়, যাহা কণার মূল ভরকে "পুনর্নির্মিত" (renormalise) করে।

দ্বিতীয় ভাগে, আমরা মূলত নন্-কমিউটেটিভ জ্যামিতির স্পেক্ট্রাল ট্রিপ্ল্ পদ্ধতি লইয়া আলোচনা করিয়াছি। নন্-কমিউটেটিভ জ্যামিতির ভিত্তিপ্রস্তর হল দেশ-কালের সাথে বীজগণিতিক তথ্যের বিনিময়। বিখ্যাত ফরাসি গণিতবিদ অ্যালেন কোন্স (Alain Connes) নন্-কমিউটেটিভ দেশ-কাল কে চিহ্নিত করার জন্য গেলফান্ড-নাইমার্ক (Gelfand-Naimark) উপপাদ্যের সাধারণীকরণ

করিয়া কমিউটেটিভ বীজগণিত কে নন্-কমিউটেটিভ C*-বীজগণিতগুলিতে প্রসারিত করিয়াছিলেন। এই তত্ত্ব অনুসারে, একটি নন্-কমিউটেটিভ জ্যামিতি বর্ণনা করার মূল উপাদান হইল একটি নন্-কমিউটেটিভ একক বীজগণিত (unital algebra) A, সেই বীজগণিতকে উপস্থাপনের উপযোগী একটি হিলবার্ট স্পেস H, এবং সীমাহীন হার্মিসীয় (unbounded hermitian) ডিরাক্ অপারেটর D। আরও কিছু সামঞ্জস্যপূর্ণ শর্ত সহকারে এই তিন বীজগাণিতিক তথ্য কে স্পেক্ট্রাল ট্রিপল বলা হয়।

- সাম্প্রতিক কালে গবেষকরা লরেন্টজিয় (অর্থাৎ ছদ্ম-রিমানিয়ান) জ্যামিতির জন্য স্পেক্ট্রাল ট্রিপল নির্মাণের চেষ্টা করিতেছেন। আমরা সেইরূপ এক ছদ্ম-রিম্যানিয়ান্ (pseudo-Riemannian) স্পেক্ট্রাল ট্রিপল পদ্ধতির ব্যবহার করিয়া ১+১ মাত্রিক লরেন্টজিয়ান ময়াল তলের জ্যামিতির জন্য স্পেক্ট্রাল ট্রিপল নির্মাণ করিয়াছি এবং তারপর দেশ-কালের মধ্যে কাল-জাতীয় (time-like) ব্যবধান রহিয়াছে এইরুপ এক জোড়া শুদ্ধাবস্থার (pure state) মধ্যে স্পেক্ট্রাল দূরত্ব গণনা করিয়াছি।

- ইহা ব্যাতীত আমরা একটি নন্-কমিউটেটিভ গোলক (fuzzy sphere S_*^2) এর জন্য, একটি রিয়াল স্ট্রাক্চার (Real structure) অপারেটর নির্মাণ করিয়াছি। এরপর দেখানো হইয়াছে যে এই স্পেক্ট্রাল ট্রিপলের ডিরাক অপারেটর, প্রথম অর্ডারের শর্তটি মানিয়া চলে না, যাহা বর্তমান স্ট্যান্ডার্ড মডেলের পরিধির ঊর্ধে কণা-পদার্থবিদ্যার কিছু নতুন সম্ভাবনার ইঙ্গিত দেয়। ইহা আমাদের ভবিষ্যতে গবেষণার একটি প্রধান লক্ষ্য।

List of publications

1. "Symmetries of κ-Minkowski space-time : A possibility of exotic momentum space geometry ?", Partha Nandi, **Anwesha Chakraborty**, Sayan Kumar Pal, Biswajit Chakraborty, Frederik G. Scholtz, *JHEP 07*(*2023*) *142*.

2. "Spectral triple with real structure on fuzzy sphere", **Anwesha Chakraborty**, Partha Nandi, Biswajit Chakraborty, *J. Math. Phys. 63* (*2022*) *2, 023504* (Selected as editor's pick).

3. "Fingerprints of the quantum space-time in time-dependent quantum mechanics: An emergent geometric phase", **Anwesha Chakraborty**, Partha Nandi, Biswajit Chakraborty, *Nucl. Phys. B 975* (*2022*) *115691*.

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This thesis is based on [1,2,3,4,5,6]

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Chapter 1
Introduction

The current understanding of particle physics would need a significant revision to effectively explain processes involving high energies of the order of Planck energy scale $\sim 10^{19}$ GeV or equivalently at a very small length scale $l_P \sim 10^{-33}$ cm. At these extreme energy levels, quantum effects of gravitational interactions are expected to play crucial role. The Standard Model (SM) of particle physics, which omits the quantum aspects of gravity, appears incomplete in this context. In fact, the energy scales involved in SM is well below Planck scale and is of the order of $\sim 10\ TeV$.
On the other hand, the General Theory of Relativity (GTR), formulated by Einstein over a century ago, is a classical field theory of gravitation. Its predictions have been subjected to rigorous experimental tests in both strong and weak gravity regimes, with increasing accuracy. Recent observations, such as the imaging of shadows of supermassive black holes and the detection of gravitational waves [1], have provided unprecedented verification of GTR's predictions. Despite these successes, there are fundamental conceptual challenges within the theory. Singularities, exemplified by black hole singularities, and the geodesic incompleteness of space-time, as demonstrated by the Hawking-Penrose singularity theorem [2, 3], point towards the limitations of GTR. In this sense, GTR implies its own limitations and suggests the need for a yet-to-be-developed theory of quantum gravity (QG).
However, a major challenge in this endeavor is the lack of experimental inputs and comprehensive description of the space-time structure at extremely small length scales (l_P). So we need a more in-depth understanding of physics at extremely tiny length scales in order to integrate gravity and quantum theory into a single theory. The convincing argument posed by Doplicher *et. al.* in [4] provided persuasive justifications for the need of a revised understanding of space-time at short length scales, which states that any effort to localize events at a scale smaller than the Planck length is thought to results in gravitational collapse. It therefore makes sense to try to incorporate this minimal length scale or maximal localization into our framework, forcing us to consider Non-commutative (NC) space-time as a possible contender of models of quantum space-time, where the space-time coordinates are elevated to operators satisfying non-vanishing NC algebra between themselves. Noncommutative geometry (NCG) ought to provide a framework to consistently couple gravity with the quantum fields of the Standard Model of particle physics.
The idea of NC space-time, however, is age-old and was first put forward by Snyder [5] in an effort to avoid the ultra-violet divergences of field theories. The development of renormalization theory drove these concepts to the sidelines until recently, when they reappeared in the quest for a coherent theory of quantum gravity. Non-commutative coordinates also appear quite naturally in low energy sector of certain string theories [6] and also in Quantum Hall effect (QHE) in condensed matter physics [7]. Specifically, spatial noncommutativity, with time regarded as an ordinary commutative parameter, has found application in various condensed matter phenomena [8–10]. For example, it has been employed to describe the physics of the anomalous quantum Hall effect [11] and the spin Hall ef-

fect [12,13]. Notably, the anomalous/spin/optical Hall phenomena have been explained by invoking Berry curvature in momentum space [14] for a semi-classical Bloch electron. Intriguingly, in such systems, the violation of time-reversal symmetry results in emergence of Berry curvature, which in turn, leads to a classical manifestation of noncommutativity among the spatial coordinates [15]. On the other hand, the quantization of Hall conductivity in the quantum Hall effect has been rigorously derived using approaches rooted in NCG [7]. In the realm of 2+1 dimensions, the NCG of the lowest Landau level assumes significance in Haldane's recent work [16] on the geometric description of the fractional quantum Hall effect. Furthermore, the consequences of coordinate noncommutativity in (2+1)D have been explored in the context of the emergence of anyonic excitation with fractional spin [17], where the NC parameter gets linked to the fractional spin parameter. The profound interplay between space-time quantization and 2+1 dimensional gravity was masterfully expounded upon by 't Hooft in [18], establishing a tight correlation between the two. This rekindled interest in NC space-time and the formulation of quantum mechanics [19–22] and quantum field theories [23, 24] on such backgrounds.

As the Planck energy scales are believed to have occurred naturally only at the very beginning of the universe, it is difficult to acquire verifiable evidence that would facilitate practicum at these energy/length scales. Non-commutative effects, however, are believed to be perceptible at high energies, temperatures, and densities, that are prevalent in very large astrophysical objects such as black holes, neutron stars, and even white dwarfs. These astrophysical objects can, perhaps, be employed as indirect sources in the quest for NC effects. Indeed, when contrasted with conventional studies, on dense fermion gases in two-dimensional non-commutative space [25] it shows significant variations in thermodynamic behavior only at high energies and low temperatures. This implies that under certain circumstances, the effects of non-commutativity could potentially be observed using current technologies. Consequently, it becomes highly captivating to construct models of quantum mechanics and quantum field theory on such NC space-times, with the aim of identifying suitable manifestations of quantum gravity (QG) physics at low-energy regimes. These manifestations can then be compared with existing theoretical models and subjected to experimental scrutiny, presenting an intriguing avenue for exploration.

Despite extensive investigations into the physical consequences of non-commutativity in quantum mechanics, quantum many-body systems [26–28,30,31], quantum electrodynamics [32–34], the standard model [35] and cosmology [36, 37], there had been a lack of a systematic formulation even for non-commutative quantum mechanics (NCQM) right at the operator level until the works by Scholtz et al. [20,21]. These studies, which presented an operator-valued formulation , are specifically tailored for two-dimensional quantum mechanics, extending the conventional quantum mechanical formalism and its associated interpretations to accommodate certain types of non-commutative (Moyal) spaces. In this formulation, the key modification lies in the utilization of more generalized Hilbert spaces that consist of Hilbert-Schmidt (HS) operators. This expanded framework allows for the description of the system's states and the determination of unitary representations within the non-commutative Heisenberg algebra. This operator-valued formulation for two-dimensional quantum mechanics and its subsequent extensions to three dimensions [29] marks a significant milestone, addressing the need for a systematic understanding of non-commutative quantum mechanics, where one confronts, so to say, the operatorial nature of space-time coordinates head-on, rather than demoting them to the usual c-numbered coordinates and compose them using star products. This helps us to evade any kind of ambiguities arising from the use of in-equivalent star products [38, 41]. This

Hilbert-Schmidt operator formulation has been used subsequently to formulate the appropriate spectral triples describing Moyal plane $\mathbb{R}^2_\theta$ and fuzzy sphere S^2_*. Further these spectral triples were used to compute spectral distances *a la'* Connes in these spaces [42].

In the above HS operator formulation of NCQM, appropriate for 2+1 and 3+1 dimensions respectively, the time wasn't treated as operators; they were just c-numbered evolution parameter as usual. The question naturally presents itself is, whether a systematic and consistent quantum mechanical formulation can be developed in the simplest Moyal type of space-time where the time $'t'$ is also an operator. There are two aspects of this formulation which are of considerable interest. Firstly, one can ask whether the presence of space-time noncommutativity with time being an operator can have an impact in a conceivable physical system. Secondly, whether the above mentioned spectral triple formulation, appropriate for spaces with Euclidean signature, can also be extended for Lorentzian signature and if so can one compute spectral distances between a pair of pure states representing a pair of events in the simplest Lorentzian space-time. In fact the answers to both the questions are in affirmative. Indeed, it has been shown in [43], that a systematic formulation of NCQM in 1+1 dimensional Moyal plane can be provided and then it can be used to show that a simple system like a forced harmonic oscillator inhabiting such a space-time allows an emergence of geometric phase, when the system is subjected to an adiabatic transport around a closed loop in the parameter space. Secondly, a spectral triple formulation, appropriate for the Lorentzian Moyal plane using HS operators can also be developed and it has been used to compute the spectral distance between pure states separated by time-like interval [44]. This is expected to be preliminary step towards the formulation of a completely noncommutative and realistic space-time which is capable of capturing some of the quantum gravity effects. This should therefore, provide an alternative prescription to the method of so called Wick rotation [45]. In fact there is a broad consensus among the practitioners of NCG and its application to the SM that the Lorentzian signature problem must be addressed head-on.

A slightly more complicated form of NC space-time is given by coordinate algebra closing under Lie brackets. This can be linked to the curved geometry of momentum space [46,47]. Intuitively, it can be related to the fact that, as the curvature in configuration/coordinate space enforces momentum-space operators to satisfy non-trivial commutator algebra so conversely the non-commuting coordinates can also be thought of as translation generators on the dual momentum space as well. Incidentally, it was Max Born, who had speculated about the necessity of curved momentum space in the context of QG way back in 1938 [48]. Later in the year 2000, it was shown again in [46, 47, 49] that the curved momentum space can be thought of as Hopf dual to NC space-time and the term 'co-gravity' was coined by the authors in this context.
In the construction of a full-fledged theory of QG, the biggest obstacle is the dearth of any experimental inputs, particularly in the high-energy regimes. Interestingly, it has been proposed quite recently in [50] that one can envisage a possible impact on the quantum origin of gravity even in the regime of weak Newtonian gravity through quantum superposition and quantum entanglement in the infrared regime [51]. This motivates us to look for systems exhibiting some plausible and robust features that may survive in an appropriately chosen regime of a future theory of QG theory. And for this, we can perhaps consider a system in a regime, where the associated length scale $l >> l_P$, but the mass scale m is comparable to $\kappa : m \leq \kappa$. With this, we are effectively considering a scenario where both $\hbar$ and G tend to zero, but their ratio $\sqrt{\hbar/G} \sim \kappa$ is held fixed. In this regime therefore, one does not have any surviving length scale in the system anymore, but rather a new mass scale $\sim \kappa$ (expected to be of the order of Planck mass m_P) emerges in the system. The presence of this mass scale can have a drastic

impact on the system dynamics, like curving the space of energy-momentum [52] or simply the momentum space , thereby deforming the relativistic kinematics and dispersion relation even for a single particle system. Since in the Lie algebraic type of non-commutativity (between the coordinates), the NC parameter has length dimensions of the order of $\sim\, l_P\,=\,\sqrt{\hbar G}$ and they effectively reduces to $\kappa^{-1}\,=\,\sqrt{\frac{G}{\hbar}}$ in the associated Poisson (or Dirac) bracket in the corresponding classical description. Thus the classical description itself can have an access to this regime of quantum gravity. In this context, we would like to mention that the well-known κ-Minkowski spacetime [47, 53–57], serves as a natural toy model which incidentally was proposed by [58] in the context of double-special relativity, where attempts were made to deform the special theory of relativity (STR) further by incorporating this new observer independent scale κ.

Several variants of κ-Minkowski space-time have been studied in the literature. Some, but not all of them, can be described by appropriate twist, which relates commutative and NC space-time (See for example [59]). In fact the kind of κ-Minkowski space-time which is being considered in this thesis (See chapter 4) is not known to admit any suitable twist. A special case of our model where the noncommutative parameter is completely 'time-like', was also considered in [60] where the momentum space was identified with AN(3) group manifold, which is a part of de-Sitter space. However, there the 4-momentum did not transform as a Lorentz 4-vector thereby deforming the Poincare algebra itself. So the question naturally arises is whether it is possible to retain the structure of Poincare ($\mathfrak{iso}(3,1)$) Lie algebra in such a manner that the space-time commutator algebra remain stable under the action of Poincare generators, but whose action is now necessarily deformed. In fact this point was investigated in [61], in a more general kind of κ-Minkowski space time involving 4 different space-time deformation parameters. It was possible then to identify the structure of the momentum space which is now parameterized by these 4 parameters. The corresponding momentum space cannot however be identified with any symmetric space-time of any kind. Not only that, such a space-time does not even have the property of a pseudo-Riemannian manifold. It is, nevertheless, possible to extract some geometrical information like geodesic distance from the distinguished point $P_\mu = 0$ to any other point on the momentum space. This in turn shows the effect of noncommutativity in the case of a simple single particle system itself through the deformed dispersion relation.

Although, the NC space-time was initially envisioned as a plausible space-time description to understand quantum gravity, Connes and his collaborators used this framework of NCG in the study of standard model of particle physics involving a scale of energy which is several orders of magnitude (~ 10 TeV) below the Planck scale, to produce a totally unified description of the Standard model of particle physics [62, 63], albeit at the classical level. This model provides a conceptual framework, which is quite successful in unifying Higgs field and other Yang-Mills gauge fields facilitating the computation of Higgs mass [64] with high precision. Somewhat in the same spirit of Kaluza Klein theory, although not considering the extra dimension, it takes into account a product space of the Euclidean commutative 4-dimensional space-time with an internal non-commutative space F in the form of $M \times F$, which are known in the literature as the Almost commutative (AC) spaces, where the non-commutativity is introduced in the framework through the space F, given typically by a matrix algebra. Note that at this energy scale one does not expect to see any effect of quantum gravity in the form of quantum space-time. It should therefore, be quite adequate to consider a commutative differential manifold M of usual type rather than any NC space to model the space-time. The choice of internal space F, described again by a "spectral triple" (algebraic data to formalize geometry in terms of algebra), is made by the requirement that it captures the gauge symmetries of the standard

model through its inner automorphisms, while the Euclidean space M captures the diffeomorphism symmetry through outer-automorphism, appropriate for the gravitational sector.
However, in spite of its various successes, the model has some shortcomings. Primary among them is that the model takes the space-time to be a 4D compact Riemannian differentiable manifold M of Euclidean signature rather than Lorentzian. Attempts are being made to circumvent this problem for quite some time now [65] like the so-called "Wick rotation" [45] etc. However, as mentioned earlier, a desirable starting point would be a formulation of Lorentzian spectral triple. There are other shortcomings in the SM itself, which till date, has not been addressed using the Connes' formulation or otherwise. For example, the SM has failed (by few percent) to unify the three gauge couplings at some high energy scale, the so called GUT scale $\sim\ 10^{16}$ GeV, which indicates that it may be necessary to add other matter couplings to change the slopes of the running of the RG equations. Furthermore, the discovery of neutrino oscillations, which imply that neutrinos do indeed have mass [66], was not initially incorporated in the SM. However, later massive neutrinos were accommodated in SM, albeit, at the expense of having to accept a number of additional free parameters in the theory. Besides, the model is able to explain neither dark matter, nor dark energy. It also cannot explain baryon asymmetry. Furthermore, the standard model has consistently defied unification with gravity. It therefore motivates us to look for some suitable avenues to explore the regimes beyond SM by suitable amendments in the existing framework of AC geometry due to Connes and his collaborators. And for that, Connes *et. al.* [67,68] themselves have shown that a possible violation in the so called first order condition may indeed open up such windows to investigate beyond SM physics. Emulating their findings we have made a modest attempt by initiating our investigation in the direction of the construction of a toy model involving SU(2) gauge theory in the framework of above mentioned Connes' AC geometry. And for that we have chosen to work with fuzzy sphere S^2_* in place of the finite space F. However, in absence of any known *even* and *real* spectral triple describing fuzzy sphere we considered the even spectral triple on S^2_* as constructed by Watamura *et. al.* [69]. We have then been able to endow this spectral triple with a real structure [70], albeit in the lowest spin-1/2 representation. We intend to carry forward this to the construction of SU(2) gauge theory as a toy model to gain some insight into the physics beyond SM.

1.1 Organization of the thesis

The thesis is organized as follows: In chapter-2, we have first reviewed two important mathematical formulation- (a) Hilbert-Schmidt operatorial formulation and (b) Spectral triple formulation using those Hilbert-Schmidt operators, which we have frequently used throughout the thesis. This thesis is based on two types of applications of NCG and thus it is divided into two parts.
Part-A, consists of chapter 3 and 4, dealing with different prototypes of NC space-time. We have then built dynamical models (both relativistic and non-relativistic) on such background. This is then followed by a discussion of its effect on dynamics of the systems.
In Chapter 3, we explored the impact of non-commutative Moyal-type spacetime in (1+1) dimensions, treating time as an operator on a *non-relativistic* time-dependent quantum mechanical system. It showcases the emergence of a geometric phase shift which is found to have its dependence on the non-commutative parameter. This chapter begins by focusing on a time-reparametrization invariant Lagrangian, which generates a non-commutative symplectic structure at the classical level. Consequently, the Lagrangian yields a constraint equation that leads to Schrödinger equation at the quantum level. We establish a formulation of NC space-time using Hilbert-Schmidt operators and define a suitable "position" basis (such as the Voros basis (2.13) built out of coherent states) to rep-

resent the Schrödinger equation in an effective commutative space-time. This provides an equivalent commutative theory incorporating NC effects. As an application, we then consider the Hamiltonian of a forced harmonic oscillator with time-varying parameters as a prototype for our non-relativistic model and derive the corresponding Schrödinger's equation. By employing a non-unitary transformation, we obtain an effective commutative description of the equation, resembling a Hamiltonian for a generalized harmonic oscillator. Furthermore, we explore the adiabatic time evolution of the system in the Heisenberg picture over a time period $\mathcal{T}$. This evolution results in a geometric phase shift, which is over and above the dynamical phase and depends explicitly on the non-commutative parameter. This phase shift serves as an indicator of effects of non-commutativity on the system.

Chapter-4 deals with a Lie-algebraic type of NC space-time, namely the κ-Minkowski space-time, where we have built up a *relativistic* model for a single massive spin-less particle.
In this chapter, our initial emphasis is on preserving Poincare symmetry of the κ-Minkowski space-time. While the Poincare algebra itself remains undeformed, the requirement of covariance of the NC space-time algebra under Poincare transformations necessitates that the actions of the algebra generators on the operator-valued space-time manifold gets suitably deformed and becomes enveloping algebra valued. We proceed by discussing the deformed co-algebraic structure of the Poincare generators, followed by a Heisenberg double construction in a Hopf-algebroid framework that serves as a consistency check for the previously derived deformed Heisenberg algebra. Subsequently, we develop a model for a spin-less relativistic massive particle that upholds the deformed Poincare symmetry. At the classical level, this model unveils a plausible regime within a potential Quantum Gravity theory, wherein the momentum space exhibits nontrivial (non-flat) geometry. Notably, in this intriguing scenario, only a mass scale (plausibly the Planck mass m_p) is present, while an associated length scale vanishes. Furthermore, we derive a deformed dispersion relation by establishing its connection with the geodesic distance in the curved momentum space. Due to the effects of noncommutativity, the mass of the particle undergoes renormalization. Importantly, we demonstrate that under specific conditions, the Planck mass can serve as an upper limit for the observed renormalized mass.

In part B, comprising of chapter 5 and 6, we concentrate on the spectral triple approach of NCG. As discussed in chapter-2, spectral triple of a NC Moyal plane in Euclidean signature can be constructed using HS operators.
In chapter-5, we have first shown how this spectral triple can be employed to compute spectral distance on Euclidean Moyal plane ($\mathbb{R}^2$) and then shown how an appropriate spectral triple can be constructed for the Lorentzian Moyal plane $\mathbb{R}^{1,1}$ and in turn can be used to compute the spectral distance between a pair of events, which are now represented by a pair of pure states, separated by time-like intervals.

In chapter-6, we have introduced the Euclidean spectral triple for a 2D fuzzy sphere S^2_* by using the novel form of the Dirac and Chirality operator describing a SU(2) covariant even spectral triple, introduced by Watamura *et. al.* in [69]. We then derived a *real structure* operator so that it can now be upgraded to a *real* and *even* spectral triple. To do so, we had to enlarge the symmetry group of the fuzzy sphere from SO(3) to O(3). This spectral triple, however, violates the *first order condition*, opening up new possibilities to further investigate some toy model of gauge theory using the above spectral triple as internal finite space and employing *almost-commutative* framework to get some insight into the physics beyond standard model.
In chapter-7, we have made some concluding remarks by summarizing the set of important results obtained in the thesis and indicated some future prospects.

Chapter 2

Brief review of Hilbert-Schmidt (HS) operatorial formulation of the spectral triple for Moyal plane ($\mathbb{R}^2_\theta$)

2.1 HS operator formulation on $\mathbb{R}^2_\theta$

In [20, 21], a HS operator-based formulation for NC quantum mechanics on Euclidean Moyal plane, was developed for the first time. It was discovered that it is mathematically structured almost in the same way as usual quantum physics and allows one to avoid the use of any star products and ambiguities arising thereof as described in chapter-1.

In non-commutative spaces coordinates are elevated to the level of operators and their commutation relations involves a NC parameter(s) and are associated with minimum length scale(s). The simplest example being the 2D Moyal plane, defined by the following commutator algebra

$$[\hat{x}_1, \hat{x}_2] = i\theta. \tag{2.1}$$

which is ISO(2) invariant algebra. This leads to the following uncertainty relation satisfied by the coordinates:

$$\Delta\hat{x}_1.\Delta\hat{x}_2 \geq \frac{\theta}{2} \tag{2.2}$$

where θ which has the dimension of squared length is the NC parameter and is taken to be related to the minimum length scale-presumably l_P if applications in QG are envisaged. The phase-space algebra remains as usual[1] given by

$$[\hat{p}_1, \hat{p}_2] = 0,\ [\hat{x}_i, \hat{p}_j] = i\delta_{ij}; \qquad i, j = 1, 2 \tag{2.3}$$

Note that we have been working with the natural unit $\hbar = 1$. Together, (2.1) and (2.3) constitute the NC Heisenberg algebra (NCHA). In order to provide a suitable representation of the coordinate algebra (2.1) we can construct the following Hilbert space $\mathcal{H}_c$, as

$$\mathcal{H}_c = Span\Big\{|n\rangle := \frac{1}{\sqrt{n}}(\hat{b}^\dagger)^n|0\rangle;\ b|0\rangle := \frac{\hat{x}_1 + i\hat{x}_2}{\sqrt{2\theta}}|0\rangle = 0\Big\} \tag{2.4}$$

[1] One can also consider non-vanishing brackets between the momentum operators [71] and a deformed phase-space bracket which naturally arises in Lie-algebraic type of noncommutative coordinates [61] (discussed in chapter-4).

After we've specified $\mathcal{H}_c$, namely the configuration space, we'll build the non-commutative quantum Hilbert space. For that we first introduce the associative NC operator algebra $(\hat{\mathcal{A}}_\theta)$ generated by $(\hat{x}_1, \hat{x}_2)$ or equivalently $(\hat{b}, \hat{b}^\dagger)$ acting on the configuration space $\mathcal{H}_c$ (2.4) as

$$\hat{\mathcal{A}}_\theta = \{|\psi) = \psi(\hat{x}_1, \hat{x}_2) = \psi(\hat{b}, \hat{b}^\dagger) = \sum_{m,n} C_{n,m}|m\rangle\langle n|\} \tag{2.5}$$

which, subject to the identification $[\hat{b}, \hat{b}^\dagger] = 1$, may be regarded as the universal enveloping algebra corresponding to (2.1) and is effectively the set of all equivalence classes of polynomials in $\hat{x}_i$ or $(\hat{b}, \hat{b}^\dagger)$ [2]. It is important to note that $\hat{\mathcal{A}}_\theta$ is not yet a Hilbert space, as is not yet equipped with any inner products or norm structures.

We can now introduce a subspace $\mathcal{H}_q \subset \hat{\mathcal{A}}_\theta$ as the space of 'Hilbert Schmidt' (HS) operators, which are bounded and compact operators with finite HS norm, and is given by,

$$\mathcal{H}_q = \left\{\psi(\hat{x}_1, \hat{x}_2) := \Big|\psi(\hat{x}_1, \hat{x}_2)\Big) \in \mathcal{B}(\mathcal{H}_c);\ \|\psi\|_{HS} := \sqrt{tr_c(\psi^\dagger\psi)} < \infty\right\} \subset \hat{\mathcal{A}}_\theta \tag{2.6}$$

where $\|\psi\|_{HS}$ is the Hilbert-Schmidt norm and tr_c denotes trace over the Hilbert space $\mathcal{H}_c$ and $\mathcal{B}(\mathcal{H}_c) \subset \hat{\mathcal{A}}_\theta$ is a set of bounded operators on $\mathcal{H}_c$. So it is worth noting that the algebra $\mathcal{A} = \mathcal{H}_q$ is a dense subspace of $\mathcal{B}(\mathcal{H}_c)$, which can be identified with a C^*-algebra, where the $*$-operation denotes hermitian conjugation, serving as an involution. The space $\mathcal{H}_q$ is equipped with the inner product

$$\Big(\psi(\hat{x}_1, \hat{x}_2), \phi(\hat{x}_1, \hat{x}_2)\Big) := tr_c\Big(\psi^\dagger(\hat{x}_1, \hat{x}_2)\phi(\hat{x}_1, \hat{x}_2)\Big) \tag{2.7}$$

and therefore is a Hilbert space. One may note at this point that, the quantum Hilbert space for the commutative QM, is the set of square-integrable coordinate functions, the appropriate generalization of which to the non-commutative scenario is the space $\mathcal{H}_q$ of Hilbert-Schmidt (HS) operators. By defining the multiplication map $\mu : \mathcal{H}_q \otimes \mathcal{H}_q \to \mathcal{H}_q$ as shown below, we may endow $\mathcal{H}_q$ with the structure of an algebra :

$$\mu\Big(|\hat{\psi}) \otimes |\hat{\phi})\Big) = |\hat{\psi}\hat{\phi}) \quad \forall\ \hat{\psi}, \hat{\phi} \in \mathcal{H}_q \tag{2.8}$$

Also, a general HS operator such as $\psi := |\psi) \in \mathcal{H}_q$, can be expanded in the Fock basis (2.4) as,

$$|\psi) = \sum_{m,n} C_{m,n}|m\rangle\langle n| \tag{2.9}$$

Now making use of the identity $|0\rangle\langle 0| =: e^{-\hat{b}^\dagger \hat{b}}$:, where the colons represent normal ordering of the operators [72], one can see that $\mathcal{A} = \mathcal{H}_q$ can be recast as a polynomial algebra generated by $\hat{b}, \hat{b}^\dagger$ as,

$$|\psi) = \sum_{m,n} C_{mn} \frac{\hat{b}^{\dagger m}}{\sqrt{m!}} |0\rangle\langle 0| \frac{\hat{b}^n}{\sqrt{n!}} = \sum_{m,n,l} C_{mn} \frac{1}{l!} \frac{(-1)^l}{\sqrt{m!n!}} (\hat{b}^\dagger)^{m+l} (\hat{b})^{n+l} \quad \in \mathcal{H}_q, \tag{2.10}$$

which is automatically in the normal-ordered form. It is important to note that $|\psi|^2_{HS} = \sum_{m,n} |C_{mn}|^2 < \infty$, which satisfies the requirement of finite norm of a Hilbert-Schmidt (HS) operator.

Now, we define the quantum space-time coordinates $(\hat{X}_i)$ and associated conjugate momentum op-

[2]The elements of $\mathcal{H}_c$ and $\mathcal{A}_\theta$ are indicated, respectively, by the angular ket $|.\rangle$ and the round ket $|.)$, as a matter of notation.

erators ($\hat{P}_i$) and their action on a generic element $\Big|\psi(\hat{x}_1,\hat{x}_2)\Big) \in \mathcal{H}_q$ is given as,

$$\hat{X}^i|\hat{\psi}) = |\hat{x}^i\hat{\psi}), \qquad \hat{P}^i|\hat{\psi}) = \frac{\epsilon^{ij}}{\theta}\Big|[\hat{x}^j,\hat{\psi}]\Big) \tag{2.11}$$

Since their domains of actions are distinct, upper case letters $\hat{X}_i$'s are used here to set them apart from their lower case counterparts $\hat{x}_i$'s : while $\hat{x}_i$'s act on $\mathcal{H}_c$, $\hat{X}_i$'s act on $\mathcal{H}_q$. However, as they adhere to the isomorphic commutator algebra, $[\hat{X}_i,\hat{X}_j] = i\theta\epsilon_{ij}$, $\hat{X}_i$ may be seen as the representation of $\hat{x}_i$. It is simple to demonstrate by using (2.11) that $(\hat{X}_i,\hat{P}_i)$ satisfy the identical NC Heisenberg algebra (2.1 ,2.3). As they act on the space of operators, these new operators might be regarded of as "super operators". In fact, they all can act on the entire $\hat{\mathcal{A}}_\theta$.

Position Basis:

As the position operators do not commute in this case, the idea of position can only be retained by employing the minimal uncertainty state, which are maximally localized through coherent states [21]. Following the example of coherent states of the Harmonic oscillator, one may propose states of least uncertainty in the classical configuration space (2.4) as

$$|z\rangle = e^{-\bar{z}\hat{b}+z\hat{b}^\dagger}|0\rangle \in \mathcal{H}_c; \qquad \text{satisfying} \quad \hat{b}|z\rangle = z|z\rangle \tag{2.12}$$

The uncertainty relation (2.2) is then saturated in these coherent states. From this we can construct a coherent state basis belonging to the $\mathcal{H}_q$ (2.6) as

$$|z,\bar{z}) := |z\rangle\langle z| \quad \text{satisfying} \quad \hat{B}|z,\bar{z}) = z|z,\bar{z}) \tag{2.13}$$

where $\hat{B} = \frac{\hat{X}^1+i\hat{X}^2}{\sqrt{2\theta}}$ is the counterpart of $\hat{b}$ operator in $\mathcal{H}_q$. The above basis satisfies the overcompleteness relation given by

$$\int \frac{d^2z}{\pi}|z,\bar{z}) \star_V (z,\bar{z}| = \mathbf{1} \tag{2.14}$$

where the outer product is composed with respect to a non-local Voros star product given by

$$\star_V := e^{\frac{i\theta}{2}\epsilon^{ij}\overleftarrow{\partial_i}\overrightarrow{\partial_j}+\frac{i\theta}{2}\overleftarrow{\partial_i}\overrightarrow{\partial_i}} \tag{2.15}$$

The associated basis $|z,\bar{z})$ or alternatively labeled as $|\vec{x})_V := \frac{1}{\sqrt{2\pi\theta}}|z,\bar{z})$ is called the Voros basis which corresponds to the position basis for Moyal plane, which satisfies

$$2\theta\int d\vec{x}\ \ |\vec{x})_V \star_V {}_V(\vec{x}| = \mathbf{1}, \qquad \text{and} \quad {}_V(\vec{x}|\vec{x}')_V = e^{-[(x_1-x_1')^2+(x_2-x_2')^2]} \tag{2.16}$$

where x_1, x_2 are the expectation values of $\hat{x}_1, \hat{x}_2$ in the coherent state basis $|z\rangle$ i.e. $\vec{x} := \langle z|\hat{\vec{x}}|z\rangle$.

We can now introduce yet another "position-like" basis in the Moyal plane, namely the Moyal basis [38]

$$|\vec{x})_M = \int \frac{d^2p}{2\pi}\, e^{-i\vec{p}.\vec{x}}\, |\vec{p}), \tag{2.17}$$

where

$$|\vec{p}) = \sqrt{\frac{\theta}{2\pi}} e^{-i\vec{p}.\hat{\vec{x}}} \tag{2.18}$$

is the eigen basis of the commuting momentum operators $\hat{P}_i$ as $\hat{P}_i\,|\vec{p}) = p_i\,|\vec{p})$ satisfying orthonormality and completeness condition as

$$(\vec{p}|\vec{p}') = \delta^2(\vec{p}-\vec{p}'), \qquad \int d^2p\;|\vec{p})\,(\vec{p}| = \mathbf{I} \tag{2.19}$$

The Moyal basis $|\vec{x})_M$ is the joint eigen-basis of commuting operators $\hat{X}^c_i$ as,

$$\hat{X}^c_i|\vec{x})_M = x_i|\vec{x})_M; \quad \text{where} \quad \hat{X}^c_i := \frac{1}{2}(\hat{X}^L_i + \hat{X}^R_i) = \hat{X}_i + \frac{\theta}{2}\epsilon_{ij}\hat{P}_j \;\text{ and }\; [\hat{X}^c_i, \hat{X}^c_j] = 0 \tag{2.20}$$

The above relation connecting commutative $\hat{X}^c_i$ to NC coordinate $\hat{X}_i$ is famously known as the 'Bopp shift' [39]. These basis also satisfy completeness condition as given by

$$\int d^2x|\vec{x})_M \star_M {}_M(\vec{x}| = \int d^2x|\vec{x})_M\; {}_M(\vec{x}| = \mathbf{1} \tag{2.21}$$

where the operation $\star_M$ is given as

$$\star_M := e^{\frac{i\theta}{2}\epsilon^{ij}\overleftarrow{\partial_i}\overrightarrow{\partial_j}} \tag{2.22}$$

Both Voros and Moyal bases gives the position bases for NC quantum mechanics, with respect to which we can take the overlap of any abstract state $\hat{\psi} \in \mathcal{H}_q$ to give its corresponding position representation or *symbols*: $\psi_{M/V}(x) = {}_{M/V}(x|\hat{\psi})$. However the representation of composite states $\hat{\psi}, \hat{\phi}$ as shown in (2.8), is given by the respective star product of two individual symbols:

$${}_{M/V}(x|\hat{\psi}\hat{\phi}) \sim {}_{M/V}(x|\hat{\psi}) \star_{M/V} {}_{M/V}(x|\hat{\phi}) \tag{2.23}$$

At this juncture it is important to point it out that, in the traditional way of analyzing quantum field theories on NC space-times, one typically demotes the aforementioned operator valued coordinates $\hat{x}_i$ to c-number valued coordinates x_i, but now composing through suitably specified star-products. The Moyal and Voros star-products are the two most used products out of a wide range of available options. The respective star-brackets yield coordinate algebras which are isomorphic to (2.1) in both the cases :

$$x^1 \star_{M/V} x^2 - x^2 \star_{M/V} x^1 = i\theta \tag{2.24}$$

Although mathematically these two star products are formally equivalent, there are, however, a number of subtle differences as well. The Moyal representation is defined on the space of functions that are of Schwartz class in configuration space, while the Voros representation is defined on a smaller subspace where it is required to be smooth at a small length scale $\sim \sqrt{\theta}$; the high momenta modes get suppressed exponentially. Note that, the map $T = e^{\frac{\theta\nabla^2}{4}}$ relating the representation of a state $\hat{\psi}$ in Moyal and Voros basis as ${}_V(\vec{x}|\hat{\psi}) = \sqrt{\frac{\theta}{2\pi}} e^{\frac{\theta\nabla^2}{4}}\, {}_M(\vec{x}|\hat{\psi})$, is not invertible on the space of square integrable or even Schwartz class functions [38].

Also, it is evident that the commuting position coordinates $\hat{X}^c_i$, of which the Moyal states (2.17) are common eigen-states of $\hat{X}^c_i$, cannot play the role of physical position observables as this will violate the minimum uncertainty in position (2.2) deriving from non-commutativity. The Moyal basis

are therefore *unphysical* in the NC quantum mechanical setting, in which case we cannot prepare the system in a Moyal state. In other words, the physical meaning of the quantum number x_i. must be different from physical position. In contrast to the Moyal basis the Voros basis does not violate the minimum uncertainty in position. Indeed, they represent the optimal spatial localization and allows the standard quantum mechanical interpretation of weak measurements [38]. From the non-commutative nature of space it is clear that the conventional interpretation of strong measurements based on projective valued measures cannot apply. Instead one must think about position measurement in the sense of a weak measurement based on a positive operator valued measure (POVM), which is satisfied by Voros basis [38]. Also note that, the Voros star product naturally occurs whenever coherent states are utilized as basis in the quantum Hilbert space. All the above discussion indicates the fact that although mathematically both the star products are equivalent at a formal level, only Voros star product and the associated Voros basis, can serve as the physical framework to describe a maximally localized states.
The physical in-equivalence between Voros and Moyal position bases on this operator formalism was demonstrated in [38], by computing the free particle transition amplitudes using the path integral formalism [38,40]. Although, these analysis were limited to the level of non-commutative quantum mechanics, similar conclusions were made in [41] in the context of QFT on non-commutative spaces, despite the fact that the position variables are no longer observables in the context of QFT and are essentially the labels of continuous degrees of freedom.

2.2 Geometry of Non-commutative space-time: Spectral triple

The fundamental concepts like points, geodesics etc. of Euclidean and Riemannian geometry do not survive in a generic NC space because of the inherent uncertainty conditions satisfied by the space-time coordinates (See for example (2.2)). In order to study the geometrical aspects of such NC spaces, we are therefore forced to take recourse to the framework of NCG, as developed by Alain Connes [73]. The study the geometry of non-commutative spaces involves, arguably, the most generalised mathematical approach to extract geometrical informations from the algebraic structures defined on the space. To start with, the well-known Gelfand Naimark theorem [74] is generalized and adapted in the framework of non-commutative geometry, which primarily involves a duality between a topological space and algebraic structure defined on that space. More concretely, according to this theorem, the category of compact Hausdorff topological spaces is dual to the category of unital commutative C^*-algebras (algebras of continuous complex valued functions). This duality between topological spaces and commutative C^*-algebras is then extended to include non-commutative C^*-algebras, and thus one can speak of the "non-commutative space" as dual to a NC C^*-algebra. We can learn about the topology of a space M by studying the C^*-algebra $\mathcal{A}$ defined on it. But to capture the geometry Connes introduced the so-called "Spectral triple" $(\mathcal{A}, \mathcal{D}, \mathcal{H})$ [75] consisting of an involutive algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ taken to be the space of normalizable spinors, through a representation π along with an unbounded, self adjoint operator $\mathcal{D}$, which is the generalized Dirac operator also acting on $\mathcal{H}$. The spectral triple with certain axioms [76] generalizes the notions of manifolds (topological spaces with additional structures) and differential geometry. Thus, a spectral triple defines a 'non-commutative manifold', where the algebra $\mathcal{A}$ provides topological information and the Dirac operator $\mathcal{D}$ provides the geometric information. In this way, one can discuss the "quantized differential calculus" known as "Non-commutative differential geometry" [75] of a quantized space using algebraic objects defined on that space. Furthermore, Connes provided a distance formula known as the "Spectral distance", between a set of pure states of the corresponding C^*-algebra $\mathcal{A}$. These pure states are the counterparts of the points in Riemannian geometry and are given by normalized positive functionals on $\mathcal{A}$. For a generic Riemannian manifold $\mathcal{M}$ the C^*-algebra $C(\mathcal{M})$ is commutative and the states are given by the Dirac's δ functional. This spectral distance generalizes the concept of geodesic distance without invoking any path. The Dirac operator $\mathcal{D}$ is required to satisfy certain conditions such as

- The operator $\mathcal{D}$ has a compact resolvant $(\mathcal{D} - \lambda)^{-1}$.
- The commutators $[\mathcal{D}, \pi(a)]\ \forall\, a \in \mathcal{A}$ are bounded, where π is some suitab;e representation of the algebra, by which it acts on $\mathcal{H}$.

With this, $\mathcal{D}^{-1}$ or, in general, $(\mathcal{D} - \lambda)^{-1}$ plays the role of infinitesimal distance and if ρ_z and ρ_w are a pair of pure states of the algebra $\mathcal{A}$ then the spectral distance between this pair of pure states is given by

$$d(\rho_z, \rho_w) = \sup_{\hat{a} \in \mathcal{B}} \{|\rho_z(\hat{a}) - \rho_w(\hat{a})|\} \tag{2.25}$$

where $\mathcal{B}$ is the ball defined as

$$\mathcal{B} = \{\hat{a} : \|[\mathcal{D}, \pi(\hat{a})]\|_{op} \leq 1\} \tag{2.26}$$

This generalizes to the conventional geodesic distance between two points on a smooth, commutative manifold.
In the context of spectral triples, an even spectral triple is characterized by the presence of a self-adjoint unitary operator known as the grading or chirality operator, denoted by γ, acting on the Hilbert space $\mathcal{H}$. This operator commutes with the representation $\pi(\mathcal{A}) : [\gamma, \pi(a)] = 0$ and anticommutes with the Dirac operator $\mathcal{D} : \{\gamma, \mathcal{D}\} = 0$. An even spectral triple of a finite space is said to be real, if there exists an antilinear isometry $\mathcal{J} : \mathcal{H} \to \mathcal{H}$, such that it maps the representation of the algebra element $\pi(a)$ to the corresponding opposite algebra representation: $\pi(a^o) = \mathcal{J}_F \pi(a^*) \mathcal{J}_F^*$. Moreover, $\mathcal{J}$ should satisfy certain properties such as $\mathcal{J}^2 = \epsilon;\ \ \mathcal{J}D = \epsilon' D\mathcal{J};\ \ \mathcal{J}\gamma = \epsilon''\gamma\mathcal{J};\ \ \ \epsilon, \epsilon', \epsilon'' \in \{1, -1\}$. The signs $\epsilon, \epsilon', \epsilon''$ depend on the KO-dimension n modulo 8 of the finite space, according to the following table [158]: While there are further axioms for a spectral triple to fully describe the topological and

n	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1		-1		1		-1	

Table 2.1: Properties of $\mathcal{J}_F$ associated with KO dimension

geometric properties of a general space, these additional axioms are not necessary for computing spectral distance. Rather, these additional structures are used in the formulation of gauge theories involving fermionic fields, particularly, in the Standard model [75]. In a spectral triple, it is the Dirac operator that provides geometric information and consequently plays a crucial role in studying the metric properties of a given commutative/NC space of Euclidean signature.
We have briefly shown below, some examples of distance computation in some simple continuous and discrete spaces such as real line $\mathbb{R}^1$ and two point space.

Distance in $\mathbb{R}^1$

The spectral triple of a 1D real line $\mathbb{R}^1$ is given by:

$$\mathcal{A} = C^\infty(\mathbb{R}^1),\ \mathcal{H} = L^2(\mathbb{R}^1),\ \mathcal{D} = -i\frac{d}{dx}$$

Action of $\mathcal{A}$ on $\mathcal{H}$ is given by $(\pi(f)\psi)(x) = f(x)\psi(x)\ \ \forall f \in \mathcal{A},\ \psi \in L^2(\mathbb{R}^1)$ is a single component spinor. Using the above spectral triple the Ball condition (2.26) reduces to

$$\|[\mathcal{D}, \pi(f)]\| = \|\frac{df}{dx}\| \leq 1$$

Pure states δ_x on $\mathcal{A}$ are defined as the evaluation map, $\delta_x(f) := f(x)$. Note that for any function $f \in \mathbb{C}^\infty(\mathbb{R}^1)$, the ball condition $\frac{df}{dx} \leq 1$ selects functions that have a slope $\frac{df}{dx}$ bounded by 1. This condition ensures that the supremum of the difference between these functions is limited by the distance between the points at which the functions are evaluated. The functions that achieve this bound are given by $f(x) = \pm x + k$, where k is a constant. Thus, the distance between two such states can be expressed as follows:

$$d(\delta_x, \delta_y) = \sup_{f\in\mathcal{A}} \left\{ |f(x) - f(y)| \,:\, \left\|\frac{df}{dx}\right\| \leq 1 \right\} = |x - y| \tag{2.27}$$

This reproduces the distance between two points x, y in $\mathbb{R}^1$.

Distance in 2-point space

Now, let's examine the simplest example of a finite space, which consists of only two points. In this space, the corresponding algebra is $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$, where the elements f are represented by pairs of complex numbers $(f(x), f(y))$, representing the evaluations of the function $f \in \mathcal{A}$ at the points x and y, respectively.

The Hilbert space is also taken to be $\mathbb{C}^2$, on which the algebra $\mathcal{A}$ acts through diagonal representation: $\pi(a)f := \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} f(x) \\ f(y) \end{pmatrix} = \begin{pmatrix} af(x) \\ af(y) \end{pmatrix}$.

The hermitian Dirac operator is given by [77]

$$\mathcal{D} = \begin{pmatrix} 0 & \Lambda \\ \bar{\Lambda} & 0 \end{pmatrix}, \quad \Lambda \in \mathbb{C}$$

The pure states representing only two points x, y in the 2-point space is represented by $\rho_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\rho_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ respectively.

The evaluation of an algebra element f on the pure states are defined by $\rho_x(f) = tr(\rho_x \pi(f)) = f(x)$ and $\rho_y(f) = tr(\rho_y \pi(f)) = f(y)$. Now the Ball condition (2.26) here reduces to $\|[\mathcal{D}, \pi(f)]\|_{op} = |f(a) - f(b)||\Lambda| \leq 1$. So the distance between two point x and y in this discrete space is given by

$$d(\rho_x, \rho_y) = \sup_{f\in\mathcal{A}} \left\{ |f(x) - f(y)| \,:\, |f(x) - f(y)||\Lambda| \leq 1 \right\} = \frac{1}{|\Lambda|} \tag{2.28}$$

which blows up if we choose $\Lambda = 0$.

2.2.1 Spectral triple for $\mathbb{R}^2_\theta$

Here we review the construction of the spectral triple $(\mathcal{A}, \mathcal{D}, \mathcal{H})$ of a Moyal plane contained in the geometric data, whose coordinate algebra is given by (2.1).
The algebra $\mathcal{A}$ of $\mathbb{R}^2_\theta$ is given by $\mathcal{H}_q$ (2.6), which forms an associative algebra (See (2.8)).
The Dirac operator for the Moyal plane is simple generalization of the $\mathbb{R}^2$ Dirac operator given by

$$\mathcal{D} = (\sigma_1 \hat{P}_1 + \sigma_2 \hat{P}_2) \tag{2.29}$$

where the adjoint action of $\hat{P}_i$'s are given as (2.11). Since it is only the Hilbert space $\mathcal{H}_q$ that furnish a complete representation of the entire Heisenberg algebra, the Dirac operator (2.29) acts on the spinorial space $\mathbb{C}^2 \otimes \mathcal{H}_q$, rendering it as the Hilbert space for the spectral triple, which is also an $\mathcal{A}$ bi-module, as it allows both left and right action of the algebra on it. Although, as we shall see in the following, that as far as the computation of distance is concerned we can take our Hilbert space to be $\mathbb{C}^2 \otimes \mathcal{H}_c$ only.
As we can see from the Ball condition (2.26) in the distance formula (2.25), we need to calculate the operatorial norm of the operator $[\mathcal{D}, \pi(a)],\ a \in \mathcal{A}$ for the calculation of distance. So let us now take a generic state $\Phi = \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \in \mathbb{C}^2 \otimes \mathcal{H}_q$ and compute the action of the above operator on it,

$$[\mathcal{D}, \pi(a)]\Phi = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & [i\hat{b}^\dagger, a] \\ [-i\hat{b}, a] & 0 \end{pmatrix} \Phi \tag{2.30}$$

where we have used a diagonal representation of the algebra element a as $\pi(a) = \begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix}$ and the action of $\mathcal{D}$ on the state Φ using (2.11), as

$$\mathcal{D}\Phi = \sqrt{\frac{2}{\theta}} \begin{pmatrix} [i\hat{b}^\dagger, \phi_2] \\ [-i\hat{b}, \phi_1] \end{pmatrix} \Phi.$$

So from (2.30), we can read of the form of the Dirac operator as

$$\mathcal{D} = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & i\hat{b}^\dagger \\ -i\hat{b} & 0 \end{pmatrix} \tag{2.31}$$

for its action on any arbitrary state Φ. However note that all actions of the algebra element a or the Dirac operator $\mathcal{D}$ on the state Φ, involves only left actions in the above computations. So our Hilbert space does not have to be a bi-module when computation of distance is concerned and we can instead work with $\mathbb{C}^2 \otimes \mathcal{H}_c$, which is a left module of the algebra $\mathcal{A} = \mathcal{H}_q$. Finally, considering the transformation $\hat{b} \rightarrow i\hat{b}$ and $\hat{b}^\dagger \rightarrow -i\hat{b}^\dagger$, which is just a SO(2) rotation by angle $\frac{\pi}{2}$ in the $x_1 - x_2$ plane (recall that our coordinate algebra (2.1) is SO(2) invariant), the Dirac operator (2.29) can be recast simply in the following form:

$$\mathcal{D} = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & \hat{b}^\dagger \\ \hat{b} & 0 \end{pmatrix} \tag{2.32}$$

which clearly has a well defined left action on $\Psi = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} \in \mathbb{C}^2 \otimes \mathcal{H}_c$. So finally, the appropriate spectral triple for Moyal plane with Euclidean signature is given by

$$\mathcal{A} = \mathcal{H}_q, \quad \mathcal{D} = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & \hat{b}^\dagger \\ \hat{b} & 0 \end{pmatrix}, \quad \mathcal{H} = \mathbb{C}^2 \otimes \mathcal{H}_c$$

Part- A

Chapter 3
Nonrelativistic system in non-commutative space-time: Emergence of geometric phase

To this day, numerous questions remain unanswered regarding the nature of time and its fundamental status in quantum theories. It is still unknown whether our current understanding of space-time is entirely consistent with both quantum mechanics and special relativity. A noticeable asymmetry exists between the treatment of space and time coordinates, with time being considered as a c-numbered parameter of evolution rather than being elevated to the level of operators, in contrast to spatial coordinates. However, to formulate a feasible theory of quantum gravity (QG) we must treat space-time at an equal footing so that the constructed model obeys at least the Lorentz symmetry. In fact, as described in chapter-1, we might need to develop such a plausible QG theory on the background of quantum space-time. In order to explore the emergence of quantum space-time, one can consider a scenario involving a superposition of two mass distributions. According to Penrose's argument in [78], this superposition, acting as feedback within the framework of Einstein's General Relativity (GR), would give rise to a superposed geometry or quantum space-time. However, it is expected that such a quantum space-time would lack time-translational symmetry, leading to inherent uncertainties in both energy (δE) and time (δt). This argument suggests the necessity of reevaluating the status of time in quantum theory and check whether the time $'t'$ should also be promoted to the level of an operator, alongside the position variables or not (as discussed in [4]). Although, the issue of time in the context of quantum mechanics itself, has been a longstanding problem in physics. In this context, one can recall the objections raised by Pauli, who observed in [79], that if time 't' were also be treated as a quantum operator, the energy spectrum would no longer be bounded from below.

In light of these considerations, it is intriguing to explore the characteristics of quantum mechanics with operator-valued time coordinates, as extensively investigated in [80–84]. Noteworthy contributions have been made by T. D. Lee [85], who demonstrated the formulation of time as a dynamical variable in non-relativistic field theories using a path-integral technique. Several previous works in this area [24, 86, 87, 89] have yielded intriguing findings, such as the discretization of time [88, 89], supporting earlier results by 't Hooft in (2+1) dimensional quantum gravity [90], as well as revisions of the spin-statistics theorem in non-commutative (NC) space-time leading to non-Pauli-like transitions [91]. Furthermore, in [92], the authors have presented an intuitive development of non-relativistic non-commutative quantum mechanics (NCQM) in 1+1 dimensional Moyal space-time. This approach requires the construction of an equivalent commutative theory, evading the Pauli's objection [79], and facilitating the exploration of concrete results in this context.

One can, at this stage, wonder about the imprint of NC nature of space-time in a low energy system as NC effects are expected to show up only at the very high energy scale. In this context, we can recall, that the very foundation of the non-relativistic quantum mechanical wave equation due to Schrödinger, lies on de Broglie's matter-wave relation $\lambda = \frac{h}{p}$ which can be rewritten in the form of $\vec{p} = \hbar\vec{k}$ and recognize that along with the Planck's relation $E = \hbar\omega$, they can be identified as the spatial and temporal components of the equation $p^{\mu} = \hbar k^{\mu}$, where p^{μ} and k^{μ} correspond to energy-momentum four vector and wave four vector respectively- transforming appropriately under Lorentz transformation. In other words, relativistic considerations have been incorporated in the construction of non-relativistic Schrödinger's wave equation. Like-wise, it is quite conceivable that perceived NC effects may be incorporated in suitable form even in some effective commutative descriptions (See for example (3.24),(3.25) in section-3.2). Besides, the construction of quantum mechanics of one particle on quantum spaces gives us a stepping stone to see the impact of noncommutativity in a simpler setting before jumping into investigations involving many particle systems or to the full fledged quantum field theoretical systems. Also, it is conceivable that there might be certain fingerprints of non-commutativity in the low-energy regime, due to incomplete decoupling between the high and low energy domains. Therefore, with this motivation in mind, in this chapter we have explored the effect of NC space-time in a non-relativistic quantum system. Particularly, we have investigated the existence of any geometric phase in a *time-dependent* dynamical system placed in a 2D Euclidean Moyal space-time. Recently, the authors of [93] has shown a link between semi-classical quantization of gravity and geometrical phase. In addition, it has been demonstrated that the state of a sensor correlated to a quantum field gains, as a result of its motion in space-time, a geometric phase that is a functional of the detector's path and is dependent upon Unruh temperature. It's interesting to note that this phase appears to fall within detectable limits at acceleration levels as small as ~ 10 m$/s^2$. Therefore, even at this tiny acceleration, one may empirically discover the Unruh effect by measuring the Berry phase[1]. This implies that the geometric phase encapsulates information also about the Unruh temperature. However, since there is still a dearth of studies investigating the relationship between the Berry phase and quantum gravity within the framework of quantum space-time, we are therefore motivated to explore the relationships, if any, between NC space time and the formation of geometrical phase in a dynamical system.

For our study, we have first recapitulated the formulation of NC quantum mechanics considering time as an operator valued coordinate [92] emulating the HS operatorial formulation as described in chapter-2, followed by the extraction of effective Schrödinger's equation in such NC space-time. As an application we then consider a forced harmonic oscillator, having periodically varying time dependent parameters, in a Moyal space-time. Following [92], we were able to obtain the effective commutative Hamiltonian of the above system which gives a generalised harmonic oscillator Hamiltonian, with perturbations that are linear in both position and momentum. Then, through the slowly changing time dependent coefficients, an adiabatic development of the overall Hamiltonian is examined. Proceeding as that of [71] we have looked at the time evolution of the ladder operators (with which the system is diagonalized) in Heisenberg's picture. We have demonstrated that the adiabatic

[1]At this point, it may be recalled that M. Berry demonstrated [94,95] that the state of the system, described by a set of time dependent parameters, when transported adiabatically around a closed loop in the parameter space, develops an additional phase factor in addition to the standard dynamical phase. It is important to emphasize once more that, Berry's phase, has no bearing on the time evolution, but rather it solely depends on the geometric nature of the parameter space.

change of the system over an entire period of time $\mathcal{T}$ leads to a total phase that includes the typical dynamical phase as well as an additional geometrical phase shift in the operator, which disappears in the commutative limit.

The chapter is organized as follows:
In section-3.1, we presented a review of the time-reparametrization invariant formulation for a non-relativistic particle in commutative space-time. At the quantum level, this formulation led to the emergence of the Schrödinger equation as a constraint equation, laying the foundation for its extension to non-commutative (NC) space-time in (1+1) dimensions, which we discuss in section-3.2.1. Moving forward to section-3.3, we considered a forced harmonic oscillator system in NC space-time and employed a coherent state basis to derive an effective Hamiltonian that describes the time evolution through an effective commutative Schrödinger equation. By performing an additional time-dependent unitary transformation, we transformed the model into a time-dependent generalized harmonic oscillator (GHO) system. The ladder operators of the GHO system were then subjected to an adiabatic time evolution over the time interval $\mathcal{T}$, leading to the generation of both the Berry phase and a dynamical phase, as discussed in section-3.3.1. Finally, in section-3.4, we present some future prospects of our work as concluding remarks.

3.1 Reparametrization symmetry and 'commutative' Quantum Mechanics

This section begins by briefly introducing the concept of a classical action that is time-reparametrization invariant. We recall the Lagrangian formalism (non-relativistic) in which the usual commutative time t is considered as configuration space variable along with the position coordinates [96] in an enlarged configuration space.
We consider the Lagrangian $L^t(x, \dot{x}, t)$ of a dynamical model in 2D and treat time t, as the zeroth component of a generalised coordinate $x^\mu : x^0 = t$. Next, we propose a new evolution parameter τ for the system's trajectory, for which we just require that, t is a monotonically increasing function of $\tau : \ t = t(\tau)$. So with the identification $dt = \dot{t}d\tau$ and $\dot{x}^\mu = \left(\frac{dx^\tau}{d\tau}\right)$ with $x^0 = t, x^1 = x$, the time-reparametrization invariant form of action [97] can be written as

$$S = \int dt\, L^t\left(x, \frac{dx}{dt}, t\right) = \int d\tau \ \dot{t}\, L^t(x, \dot{x}, t, \dot{t}) = \int \ d\tau \ L^\tau(x^\mu, \dot{x}^\mu) \tag{3.1}$$

This action is invariant under the world line reparametrization $\tau \to \tau' = \tau'(\tau), \quad$ with $x^\mu(\tau)$ transforming as world-line scalars: $x^\mu(\tau) \to x^{'\mu}(\tau') = x^\mu(\tau)$. Note that $L^\tau(x^\mu, \dot{x}^\mu)$ in (3.1) is our modified Lagrangian. As an example, let us consider the Lagrangian of a non-relativistic particle in presence of a generic potential $V(x, t)$:

$$L^t(x, \dot{x}, t) = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - V(x, t) \tag{3.2}$$

Considering the evolution parameter to be τ , the corresponding modified Lagrangian of the theory is given by,

$$L^\tau = \frac{1}{2}m\frac{\dot{x}^2}{\dot{t}} - \dot{t}V(x, t) \tag{3.3}$$

The corresponding canonical momenta are given by,

$$p_x = \frac{dL^\tau}{d\dot{x}} = m\frac{\dot{x}}{\dot{t}} = m\frac{dx}{dt}$$
$$p_t = \frac{dL^\tau}{d\dot{t}} = -\frac{m}{2}\left(\frac{dx}{dt}\right)^2 - V(x,t) = -\frac{p_x^2}{2m} - V(x,t) = -H \tag{3.4}$$

where $H = \frac{p_x^2}{2m} + V(x,t)$ is the canonical Hamiltonian corresponding to (3.2). There exists a primary constraint in the theory as can be seen from (3.4) and is given by

$$\phi = p_t + H \approx 0 \tag{3.5}$$

where the notation $\approx$ is used to denote weak equality a *la* Dirac [97, 98]. The Hamiltonian corresponding to the modified Lagrangian L^τ (3.3) is proportional to ϕ (3.5) and therefore also vanishes weakly:

$$H^\tau = p_t\dot{t} + p_x\dot{x} - L^\tau = \dot{t}\phi \approx 0 \tag{3.6}$$

So it is evident that the system has a first class constraint ϕ. The total Hamiltonian for constrained theory according to Dirac's prescription is then given by

$$H_T^\tau = H^\tau + \sigma(\tau)\phi = \sigma(\tau)\phi \tag{3.7}$$

where $\sigma(\tau)$ is the Lagrange's multiplier. Now we can perform the inverse Legendre transformation and write the Lagrangian in its first order form as

$$L_f^\tau = p_\mu \dot{x}^\mu - \sigma(\tau)(p_t + H), \qquad \mu = 0, 1 \tag{3.8}$$

It is worth noting that in the first order formalism, the configuration space is expanded further by including the canonical momenta as generalised coordinates. We can now simply perform the procedure of Dirac's constraint analysis [97,98] for the above Lagrangian (3.8) to discover one major first class constraint and two pairs of primary second class constraints. We then strongly implement the system's second class constraints by introducing Dirac's brackets (DB), which are given by

$$\{x^\mu, x^\nu\}_D = 0 = \{p_\mu, p_\nu\}_D; \quad \{x^\mu, p_\nu\}_D = \delta^\mu{}_\nu \tag{3.9}$$

Now one can easily carry out canonical quantization after elevating the phase space variables (t, x, p_t, p_x) to the level of operators and the Dirac brackets to the status of commutator brackets fulfilling

$$[\hat{t}, \hat{x}] = 0 = [\hat{p}_t, \hat{p}_x], \qquad [\hat{t}, \hat{p}_t] = i = [\hat{x}, \hat{p}_x] \ \ (\hbar = 1) \tag{3.10}$$

We then seek for a suitable Hilbert space on which these operators act and therefore furnishes an appropriate representation of this algebra. The quantum analogue of the 2D configuration space is now the Hilbert space $L^2(\mathbb{R}^2)$. We can here provide the simultaneous "spatio-temporal" eigenbasis $|x, t\rangle$ of the commutating operators $\hat{t}$ and $\hat{x}$ fulfilling

$$\hat{t}\,|x,t\rangle = t\,|x,t\rangle\,, \quad \hat{x}\,|x,t\rangle = x\,|x,t\rangle\,. \tag{3.11}$$

which satisfy the completeness and orthonormality condition as follows :

$$\int dtdx\, |x,t\rangle \langle x,t| = \mathbb{I}, \quad \langle x,t|t^{'},x^{'}\rangle = \delta(t-t^{'})\delta(x-x^{'}) \tag{3.12}$$

The coordinate representations of phase space operators are given by

$$\begin{aligned} \langle x,t|\hat{x}|\psi\rangle = x\,\langle x,t|\psi\rangle\,, \langle x,t|\hat{t}|\psi\rangle = t\,\langle x,t|\psi\rangle \\ \langle x,t|\hat{p}_x|\psi\rangle = -i\partial_x\,\langle x,t|\psi\rangle\,, \langle x,t|\hat{p}_t|\psi\rangle = -i\partial_t\,\langle x,t|\psi\rangle \end{aligned} \tag{3.13}$$

where $\psi(x,t) = \langle x,t|\psi\rangle \in L^2(\mathbb{R}^2)$ satisfies the following square-integrability condition:

$$\langle\psi|\psi\rangle = \int dtdx\; \psi^*(x,t)\psi(x,t) < \infty \tag{3.14}$$

The associated inner product is given by

$$<\psi|\phi>_= \int dtdx\psi^\star(x,t)\phi(x,t) \tag{3.15}$$

However, in order to get a suitable probabilistic interpretation of non-relativistic QM, we need to isolate the physical Hilbert space, which is the kernel of $\hat{\phi}$ i.e.

$$\hat{\phi}\,|\psi_{phy}\rangle = (\hat{p}_t + \hat{H})\,|\psi_{phy}\rangle = 0, \tag{3.16}$$

where $\hat{\phi}$ is the operator form of the first class constraint (3.5) [96]. This is equivalent to requiring that physical states be gauge invariant.
The temporal evolution of the physical states or wave-functions ($< x,t|\psi_{phy} >:= \psi_{phy}(x,t)$) is easily obtained from the coordinate representation of the quantum constraint equation (3.16) as

$$i\frac{\partial}{\partial t}\psi_{phy}(x,t) = \left(-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right)\psi_{phy}(x,t) \tag{3.17}$$

which is what we refer to as the time-dependent Schrödinger equation. It is important to note that it is independent of the parameter τ since its τ-evolution, which is nothing more than the unfolding of the gauge transformation caused by the first class constraint ϕ (3.5), has been frozen, as can be seen by using (3.6, 3.16).
In order to get the probabilistic interpretation of this Schrödinger equation, we must first note that the corresponding continuity equation, given by

$$\frac{\partial\rho}{\partial t} + \frac{\partial J_x}{\partial x} = 0 \tag{3.18}$$

with $\rho = \psi^\star_{phy}(x,t)\psi_{phy}(x,t)$ and $J_x = \frac{1}{2m}Im(\psi^\star_{phy}(x,t)\overleftrightarrow{\partial_x}\psi_{phy}(x,t))$. Correspondingly, after performing a spatial integration in both sides of (3.18) we can write,

$$\partial_t\int_{-\infty}^{\infty}\rho dx = -\int_{-\infty}^{\infty}(\partial_x J_x)dx = 0 \tag{3.19}$$

where we have leveraged the fact that our physical wave-function satisfies $\psi_{phy}(x,t) \to 0$ as $x \to \pm\infty$, as opposed to (3.14) where we additionally demand $\psi_{phy}(x,t) \to 0$ as $t \to \pm\infty$, which obviously will not be acceptable for a probabilistic interpretation. As a result, the non-negative ρ may be read as a probability density, and this condition of physicality of state defined on a "space-like surface" shows the conservation of the total probability $\int_{-\infty}^{\infty} \rho dx$. The variable t now serves as the evolution parameter, as may be recognised from the statement of the continuity equation (3.18). As a result, in order for $\psi_{phy}(x,t)$ to be "well behaved," it must be normalizable and meet the following square integrability criterion at constant time slice

$$\langle\psi_{phy}|\psi_{phy}\rangle_t = \int_{-\infty}^{\infty} dx\ \psi^*_{phy}(x,t)\psi_{phy}(x,t) < \infty \tag{3.20}$$

In other words, the physical states fulfilling (3.17) cannot be the elements of the Hilbert space $L^2(\mathbb{R}^2)$. We will, however, limit our focus to physical states, and henceforth we will no longer use "phy" in the subscript and will write it just as $\psi(x,t)$ for brevity, except when needed explicitly. So the probabilistic interpretation is recovered by substituting the inner-product (3.15) with one that simply includes a spatial integration at a fixed time slice:

$$\langle\psi|\phi\rangle_t := \int_t dx\ \psi^*(x,t)\phi(x,t). \tag{3.21}$$

This will be referred as "induced inner product". It is evident that , states satisfying the inner product of $L^2(\mathbb{R}^1)$ space (3.21) might not be an element of $L^2(\mathbb{R}^2)$ (3.15).

For example, let us consider a stationary state of the form $\psi_E(x,t) = e^{-iEt}\psi(x)$ with energy E. Evidently, $\psi_E(x,t) \in L^2(\mathbb{R}^1)$ but $\psi_E(x,t) \notin L^2(\mathbb{R}^2)$. But note that this state can be expressed as a superposition of plane waves with various momentum p as $\psi_E(x,t) = \int dp\, e^{-i(Et-px)}\tilde{\psi}(p)$, where the exponential $\frac{1}{2\pi}e^{-i(Et-px)} := \langle x,t|E,p\rangle$ is the overlap of simultaneous eigenstate $|E,p\rangle$ of both $\hat{p}_t$ and $\hat{p}_x$ with the basis $|x,t\rangle$ (3.11). Now considering a pair of such stationary states $|E,p\rangle$ and $|E',p'\rangle$, one gets, using the inner product (3.15),

$$\langle E',p'|E,p\rangle = \delta(E'-E)\,\delta(p'-p) \tag{3.22}$$

This proves that for same energy eigenvalues i.e. $E' = E$, the right hand side in (3.22) diverges. On the other hand, if we use the induced inner product (3.21), it yields

$$\langle E,p'|E,p\rangle_t = \frac{1}{2\pi}\delta(p'-p) \tag{3.23}$$

It is nothing but the standard orthonormality condition involving momentum states that is devoid of such divergence. As we will see in the following section that an identical situation exists in our type of NC space-time. In either scenario, we will primarily work with this induced inner product, or more broadly between pairs of wave packets such as $|\psi\rangle$ and $|\phi\rangle$ (3.21), in order to retrieve the probabilistic interpretation using this inner product of $L^2(\mathbb{R}^1)$.

Finally, observe that the self-adjoint property of the derivative representation of $\hat{p}_t = -i\partial_t$ in (3.13) no longer holds in the Hilbert space $L^2(\mathbb{R}^1)$ with associated inner product (3.21) because it is not possible to demand that $|\ \psi(x,t)\ | \to 0$ as $|\ t\ | \to \infty$ as mentioned above, which is needed in $L^2(\mathbb{R}^2)$, to perform integration by parts with respect to $'t'$ and drop boundary term. As a result, we will exclude $\hat{t}$ and $\hat{p}_t$ from the dynamical phase-space variables. The former is now associated with the new evolution parameter after being "demoted" to a c-number parameter and losing its link with $\hat{p}_t$. As a result, (3.17) is now considered a postulate.

3.2 Quantum mechanics on non-commutative space-time

The analysis of a typical time-independent system in NC space can result in some changes of system's dynamics, as was demonstrated earlier in [38]. For example, modifications are made to the form of the system's propagator and the wave function of the system is also deformed. However, when such a system is put in the background of a NC space-time (i.e., where time also functions as an operator), the system dynamics yields the same result as that of commutative system and no NC correction is obtained in the Hamiltonian and therefore in the spectrum of the system (Appendix-A). This encourages us to investigate a time-*dependent* system living in an NC space-time and look for NC effects, if any.

Here we take into account the time-dependent Hamiltonian of a forced harmonic oscillator (FHO), which is well known to have several uses in quantum optics. The main inspiration for this was the Hamiltonian structure presented by Mehta and Sudarshan [99], which demonstrates that coherent states continue to remain coherent during time development if the Hamiltonian's general form is required to be that of an FHO. Authors have used a similar Hamiltonian to demonstrate that an FHO coupled to a transient classical force may be simply characterised in terms of coherent states in [100], which is highly helpful in characterizing the light from optical masers. Therefore, finding the above-mentioned signature in the form of a geometric phase in such a system will be an interesting problem to investigate. To do this, however, we must first establish the formalism of quantum mechanics (QM) in which time is an operator in NC space and time. In this section, we're going to discuss about it. Before that, let us begin with a simple example to demonstrate how the NC configuration space may appear in a typical classical dynamical system.

To achieve our goals, we can modify the first order Lagrangian L_f^τ (3.8) by adding a Chern-Simon (CS) type term in the momentum space [101]. This is inspired by [102, 103], where it was demonstrated that the phase-space representation of action of a particle positioned in an NC plane in the presence of an arbitrary potential leads to the emergence of an additional "exotic" term in action, which is a Chern-Simon (CS) like term in momentum space, and the corresponding CS parameter is essentially the source of noncommutativity. The Galilean symmetry of the system is unaffected by this inclusion of the CS term in the Lagrangian. [2]
So let's consider the following Lagrangian in a classical environment to develop an NC extension of the commutative space-time,

$$L_f^{\tau,\theta} = p_\mu \dot{x}^\mu + \frac{\theta}{2}\epsilon^{\mu\nu} p_\mu \dot{p}_\nu - \sigma(\tau)(p_t + H), \quad \mu, \nu = 0, 1 \tag{3.24}$$

where $p_\mu = (p_t, p_x)$. Running the Dirac's constraint analysis (Appendix-B) on the above system we get the following Dirac brackets between the phase space variables.

$$\{x^\mu, x^\nu\}_D = \theta\epsilon^{\mu\nu}; \quad \{p_\mu, p_\nu\}_D = 0; \quad \{x^\mu, p_\nu\}_D = \delta^\mu{}_\nu \tag{3.25}$$

[2] For a Galilean transformation $\delta x = vt, \delta t = 0$, we obtain the with the following momentum transformation as $\delta p_x = mv$, $\delta p_t = -\delta H_{free} = -p_x v$, where H_{free} is the free particle Hamiltonian. This transformation makes it simple to demonstrate that the Chern-Simon term $\epsilon^{\mu\nu} p_\mu \dot{p}_\nu$ changes up to a total derivative that may be disregarded in the action. Therefore, under Galilean transformation, the CS action is quasi-invariant.

Thus our classical system (3.24) naturally gives rise to a configuration space that has NC character which manifests through a deformed symplectic structure (3.25).
We shall now promote the classical brackets (3.25) to the level of commutation brackets in order to describe non-relativistic QM in NC space-time as,

$$[\hat{t}, \hat{x}] = i\theta, \tag{3.26}$$

with θ being the NC parameter, along with

$$[\hat{p}_t, \hat{p}_x] = 0, \; [\hat{t}, \hat{p}_t] = i = [\hat{x}, \hat{p}_x]. \tag{3.27}$$

Note that we have been working with the natural unit $\hbar = 1$. Together, (3.26) and (3.27) constitute the NC Heisenberg algebra (NCHA).
In reference to section-2.1 in chapter-2, we can now construct the Hilbert space in a similar manner (See (2.4)), which furnishes a representation for the NC coordinate sub-algebra in (3.26) as [20, 21, 71, 92],

$$\mathcal{H}_c = Span\left\{ |n\rangle = \frac{(\hat{b}^\dagger)^n}{\sqrt{n!}}|0\rangle; \; \hat{b}|0\rangle = \frac{\hat{t} + i\hat{x}}{\sqrt{2\theta}}|0\rangle = 0 \right\} \tag{3.28}$$

We now introduce the associative NC operator algebra ($\hat{\mathcal{A}}_\theta$) generated by $(\hat{t}, \hat{x})$ or equivalently $(\hat{b}, \hat{b}^\dagger)$ acting on the configuration space $\mathcal{H}_c$ (3.28) as

$$\hat{\mathcal{A}}_\theta = \{|\psi) = \psi(\hat{t}, \hat{x}) = \psi(\hat{b}, \hat{b}^\dagger) = \sum_{m,n} c_{n,m}|m\rangle\langle n|\} \tag{3.29}$$

which is not a Hilbert space yet as was discussed in chapter-2, as it does not posses any inner-product structures. We now introduce a subspace $\mathcal{H}_q$ of $\hat{\mathcal{A}}_\theta$, acting again on $\mathcal{H}_c$ (3.28) given by,

$$\mathcal{H}_q = \left\{ \psi(\hat{t}, \hat{x}) \equiv \Big|\psi(\hat{t}, \hat{x})\Big) \in \mathcal{B}(\mathcal{H}_c); \; \|\psi\|_{HS} := \sqrt{tr_c(\psi^\dagger \psi)} < \infty \right\} \subset \hat{\mathcal{A}}_\theta \tag{3.30}$$

where $\|\psi\|_{HS}$ is the Hilbert-Schmidt norm and tr_c denotes trace over the Hilbert space $\mathcal{H}_c$ and $\mathcal{B}(\mathcal{H}_c) \subset \hat{\mathcal{A}}_\theta$ is a set of bounded operators acting on $\mathcal{H}_c$. This space $\mathcal{H}_q$ is equipped with the inner product

$$\Big(\psi(\hat{t}, \hat{x}), \phi(\hat{t}, \hat{x})\Big) := tr_c\Big(\psi^\dagger(\hat{t}, \hat{x})\phi(\hat{t}, \hat{x})\Big) \tag{3.31}$$

and therefore forms a Hilbert space-the space of HS operators. Now, we define the quantum space-time coordinates $(\hat{T}, \hat{X})$ and associated conjugate momentum operators $(\hat{P}_t, \hat{P}_x)$ and their respective actions on a generic element $\Big|\psi(\hat{t}, \hat{x})\Big) \in \mathcal{H}_q$ is given as,

$$\begin{aligned} \hat{T}\Big|\psi(\hat{t}, \hat{x})\Big) &= \Big|\hat{t}\psi(\hat{t}, \hat{x})\Big), \quad \hat{X}\Big|\psi(\hat{t}, \hat{x})\Big) = \Big|\hat{x}\psi(\hat{t}, \hat{x})\Big), \\ \hat{P}_x\Big|\psi(\hat{t}, \hat{x})\Big) &= -\frac{1}{\theta}\Big|[\hat{t}, \psi(\hat{t}, \hat{x})]\Big), \;\; \hat{P}_t\Big|\psi(\hat{t}, \hat{x})\Big) = \frac{1}{\theta}\Big|[\hat{x}, \psi(\hat{t}, \hat{x})]\Big) \end{aligned} \tag{3.32}$$

It can be checked using (3.32) that the algebra satisfied by $\{\hat{X}, \hat{T}, \hat{P}_x, \hat{P}_t\}$ is isomorphic to (3.26) and (3.27). Note that the HS norm (3.30) is the NC counterpart of (3.14). Again, it is obvious that in order to have an appropriate probabilistic interpretation, the corresponding inner-product (3.31), much like its counterpart (3.15) will not be suitable one. Rather, we should have an inner product defined on a constant time slice, similar to (3.21). In the next section, we address it.

3.2.1 Schrödinger equation and an induced inner product

It is now obvious that we cannot identify a counterpart to the common space-time eigenstate $|x,t\rangle$ (3.11) in light of $\theta \neq 0$. Nevertheless, by utilizing the coherent state, we may still obtain an effective commutative theory, as we shall demonstrate below. In order to do so, we take the Sudarshan-Glauber coherent state belonging to $\mathcal{H}_c$ (3.28) exactly as (2.12) in chapter-2:

$$|z\rangle = e^{-\bar{z}\hat{b}+z\hat{b}^\dagger}|0\rangle \in \mathcal{H}_c. \tag{3.33}$$

This is an eigen state of the annihilation operator : $\hat{b}|z\rangle = z|z\rangle$, where $\hat{b} = \frac{\hat{t}+i\hat{x}}{\sqrt{2\theta}}$ and z is dimensionless complex number and is given by,

$$z = \frac{t+ix}{\sqrt{2\theta}}; \qquad t = \langle z|\hat{t}|z\rangle, x = \langle z|\hat{x}|z\rangle \tag{3.34}$$

Here t and x are effective commutative coordinate variables. We can now construct the counterpart of coherent state basis in $\mathcal{H}_q$ (3.30), by taking the outer product of the bases $|z\rangle \equiv |x,t\rangle$ (3.33), as

$$|z,\bar{z}) \equiv |z) = |z\rangle\langle z| = \sqrt{2\pi\theta}\ |x,t) \in \mathcal{H}_q \quad \text{fulfilling} \;\; B|z) = z|z) \tag{3.35}$$

where the operator $\hat{b} \in \hat{\mathcal{A}}_\theta$ is represented by the annihilation operator $\hat{B} = \frac{\hat{T}+i\hat{X}}{\sqrt{2\theta}}$ (3.30). Note that in this scenario, $|z) \in \mathcal{H}_q$ (3.35) saturates the space-time uncertainty: $\Delta\hat{T}.\Delta\hat{X} = \frac{\theta}{2}$. This implies that such a state represents a maximally localized "point" or rather an event in space-time. In fact, because it is a pure density matrix, this state can be thought of as a pure state of the algebra $\hat{\mathcal{A}}_\theta$ (3.29). It is a counterpart of a point in the corresponding commutative C^*-algebra $C^\infty(\mathbb{R}^2)$ that describes the (1+1) D commutative plane as represented by Dirac's delta functional. The basis $|z,\bar{z}) \equiv |z)$ satisfies the over-completeness property:

$$\int \frac{d^2z}{\pi}|z,\bar{z}) \star_V (z,\bar{z}| = \int dtdx\,|x,t) \star_V (x,t| = \mathbf{1}_q, \tag{3.36}$$

where $*_V$ represents the Voros star product and is given by,

$$\star_V = e^{\overleftarrow{\partial_z}\overrightarrow{\partial_{\bar{z}}}} = e^{\frac{i\theta}{2}(-i\delta_{ij}+\epsilon_{ij})\overleftarrow{\partial_i}\overrightarrow{\partial_j}}; \quad i,j=0,1; \;\; x^0=t, x^1=x; \quad \epsilon_{01}=1 \tag{3.37}$$

Similar to formal QM, the coherent state representation of an abstract state $\psi(\hat{t},\hat{x})$ yields, by using (3.31),(3.35) the standard coordinate representation of a state:

$$\psi(x,t) := (x,t|\psi(\hat{x},\hat{t})) = \frac{1}{\sqrt{2\pi\theta}}\Big(z,\bar{z}|\psi(\hat{x},\hat{t})\Big) = \frac{1}{\sqrt{2\pi\theta}}tr_c\Big[|z\rangle\langle z|\psi(\hat{x},\hat{t})\Big] = \frac{1}{\sqrt{2\pi\theta}}\langle z|\psi(\hat{x},\hat{t})|z\rangle \tag{3.38}$$

and is called the *symbol* of the HS operator $\psi(\hat{x},\hat{t})$. The analogous symbol of a composite operator, such as $\psi(\hat{x},\hat{t})\phi(\hat{x},\hat{t})$, is obtained by composing the respective symbols through the Voros star product:

$$\Big(z\Big|\psi(\hat{x},\hat{t})\phi(\hat{x},\hat{t})\Big) = \Big(z\Big|\psi(\hat{x},\hat{t})\Big) \star_V \Big(z\Big|\phi(\hat{x},\hat{t})\Big) \tag{3.39}$$

As a result, the space of HS operators $\mathcal{H}_q$ and the space of their corresponding symbols are now isomorphic. Whereas the composition rule in the former case is determined by the normal product of operators, the composition rule in the latter case is determined by the Voros star product. The overlap of two generic states $(|\psi), |\phi))$ in the quantum Hilbert space $\mathcal{H}_q$ may now be expressed using (3.36) by the form

$$(\psi|\phi) = \int dtdx\ \psi^*(x,t) \star_V \phi(x,t) \tag{3.40}$$

We now introduce the coordinate representation of the phase space operators to derive the effective commutative Schrödinger equation in coordinate space. To begin with, it should be noted that the coherent state representation of the left actions of the space-time operators $\hat{X}_L, \hat{T}_L$, operating on any arbitrary element $|\psi)$ in $\mathcal{H}_q$, may be expressed by using (3.39) as,

$$\left(x,t\middle|\hat{X}_L\,\psi(\hat{x},\hat{t})\right) = \frac{1}{\sqrt{2\pi\theta}}\left(z,\bar{z}\middle|\hat{x}\psi\right) = \frac{1}{\sqrt{2\pi\theta}}\ \langle z|\hat{x}|z\rangle \star_V (z,\bar{z}|\psi(\hat{x},\hat{t})) \tag{3.41}$$

Finally making use of (3.38) this readily yields

$$\left(x,t|\hat{X}_L\,\psi(\hat{x},\hat{t})\right) = X^L_\theta \left(x,t|\psi(\hat{x},\hat{t})\right) = X^L_\theta\ \psi(x,t) \tag{3.42}$$

with

$$X^L_\theta = \left[x + \frac{\theta}{2}(\partial_x - i\partial_t)\right] \tag{3.43}$$

Proceeding exactly in the same way, we obtain the representation of $\hat{T}$ as

$$T^L_\theta = \left[t + \frac{\theta}{2}(\partial_t + i\partial_x)\right], \tag{3.44}$$

The operators X_θ and T_θ are defined in such a way that they satisfy the commutation relation $[T^L_\theta, X^L_\theta] = i\theta$ when acting on arbitrary states. By considering an arbitrary pair of states $|\psi_1)$ and $|\psi_2)$ in $\mathcal{H}_q$ along with their associated symbols, it can be proven that both X_θ and T_θ are self-adjoint with respect to the inner product (3.40) (refer to Appendix-C) by exploiting the associativity of the Voros star product. The property of self-adjointness for X_θ and T_θ will also hold for the "induced" inner product, which will be introduced below (3.56) as the counterpart of (3.21) as the analysis is independent of the integration measure.

In a similar vein, the right action of $\hat{X}$ and $\hat{T}$ on $|\psi)$ is given by

$$\hat{X}_R|\psi) = \left|\psi(\hat{x},\hat{t})\hat{x}\right); \quad \hat{T}_R|\psi) = \left|\psi(\hat{x},\hat{t})\hat{t}\right) \tag{3.45}$$

where the corresponding representations are obtained as,

$$X^R_\theta = \left[x + \frac{\theta}{2}(\partial_x + i\partial_t)\right], \quad T^R_\theta = \left[t + \frac{\theta}{2}(\partial_t - i\partial_x)\right] \tag{3.46}$$

satisfying $[T^R_\theta, X^R_\theta] = -i\theta$, which is suitable for the opposite algebra generators belonging to $\hat{\mathcal{A}}^o_\theta$. Finally, while discussing the coherent state representation of momenta operators in (3.32), we note that the adjoint actions of momenta operators may be expressed equivalently as the differences between left and right actions of position operator, which results in,

$$\Big(x,t|\hat{P}_t\psi(\hat{x},\hat{t})\Big) = -i\partial_t\psi(x,t)\ ;\ \ \Big(x,t|\hat{P}_x\psi(\hat{x},\hat{t})\Big) = -i\partial_x\psi(x,t) \tag{3.47}$$

We now observe that our NC theory (3.24) is endowed with a supplementary first-class constraint $\phi \approx 0$, which allows us to isolate the physical Hilbert space $\mathcal{H}_{ph} \subset \hat{\mathcal{A}}_\theta$ (see Appendix-B). Therefore, we must impose the requirement that the physical states $|\psi_{phy}) = \psi_{phy}(\hat{x},\hat{t})$ are annihilated by the operatorial form of this constraint ϕ in order to achieve an effective commutative Schrödinger equation in NC space-time:

$$(\hat{P}_t + \hat{H})|\psi_{phy}) = 0; \qquad \psi_{phy}(\hat{x},\hat{t}) \in \hat{\mathcal{A}}_\theta \tag{3.48}$$

where $\hat{H} = \frac{\hat{P}_x^2}{2m} + V(\hat{X},\hat{T})$. This is the counterpart of the commutative case (3.16). Now we can take the overlap of (3.48) with $|x,t)$ basis (3.35), to obtain the effective commutative time-dependent Schrödinger equation in quantum space-time. Using (3.43,3.44,3.47) we finally get,

$$i\partial_t\psi_{phy}(x,t) = \left[-\frac{1}{2m}\partial_x^2 + V(x,t)\star_V\right]\psi_{phy}(x,t) \tag{3.49}$$

Taking complex conjugate of the above equation we get

$$-i\partial_t\psi^*_{phy}(x,t) = -\frac{1}{2m}\partial_x^2\psi^*_{phy}(x,t) + \psi^*_{phy}(x,t)\ \star_V\ V(x,t). \tag{3.50}$$

Now using (3.49) and (3.50) one therefore obtains the continuity equation

$$\partial_t\rho_\theta + \partial_x J^x_\theta = 0 \tag{3.51}$$

where

$$\begin{aligned}\rho_\theta &= \psi^*_{phy}(x,t)\ \star_V\ \psi_{phy}(x,t),\\ J^x_\theta &= \frac{1}{2im}[\psi^*_{phy}\star_V(\partial_x\psi_{phy}) - (\partial_x\psi^*_{phy})\star_V\psi_{phy}].\end{aligned} \tag{3.52}$$

A spatial integration over x on both sides of (3.51) over the entire range of $\mathbb{R}^1$ now yields,

$$\partial_t\int_{-\infty}^{\infty}\rho_\theta dx = -\int_{-\infty}^{\infty}(\partial_x J^\theta_\theta)dx = 0 \tag{3.53}$$

where the condition $\psi_{phy}(x,t), \to 0$ as $x \to \pm\infty$ has been imposed which means $\psi_{phy}(x,t) \in L^1_*(\mathbb{R}^1)$ is square integrable [3], so that right hand side of the above equation becomes zero giving the conservation of the total probability $\int_{-\infty}^{\infty}\rho_\theta dx$ whereby we can write $\rho_\theta(x,t)$ in a manifestly positive definite form.

$$\rho_\theta(x,t) = \psi^*_{phy}(x,t)\ \star_V\ \psi_{phy}(x,t) = \frac{1}{2\pi\theta}\psi^*_{phy}(z,\bar{z})\star_V\psi_{phy}(z,\bar{z}) = \frac{1}{2\pi\theta}\sum_{n=0}^{\infty}\frac{1}{n!}|\partial_z^n\psi_{phy}(z,\bar{z})|^2 > 0. \tag{3.54}$$

This implies that $\rho_\theta(x,t)$, may be interpreted as the probability density at a particular point in time. The variable t here serves as the evolution parameter, as shown by the continuity equation (3.51), and

[3]It may be noted that $L^2_\star(\mathbb{R}^1)$ involves θ-deformed $\star$ multiplication rule, in contrast to $L^2(\mathbb{R}^1)$ which involves just the usual point-wise multiplication.

for this reason we need $\psi_{phy}(x,t)$ to be "well behaved," or to satisfy the following square integrability criterion at a constant time slice:

$$\langle\psi_{phy}|\psi_{phy}\rangle_{*\,t} = \int_{-\infty}^{\infty} dx\ \psi^*_{phy}(x,t) \star_V \psi_{phy}(x,t) < \infty, \tag{3.55}$$

Consequently, $\psi_{phy}(x,t) \in L^2_\star(\mathbb{R}^1)$, which is substantially different from $L^2_\star(\mathbb{R}^2)$. These are all simple generalizations of the corresponding commutative case discussed in section-3.1 (See, particularly, below (3.21)). Clearly, a spatial integration over x for a specific time slice should also define the associated inner-product for the space as follows:

$$\langle\psi_{phy}|\phi_{phy}\rangle_{*,\,t} = \int_{-\infty}^{\infty} dx\ \psi^*_{phy}(x,t) \star_V \phi_{phy}(x,t) < \infty. \tag{3.56}$$

We must reiterate at this stage that, here t and x should not be considered time and space coordinates; rather, they are just the coherent state expectation values, as defined in (3.34).
It therefore becomes clear that the physical states satisfying Schrödinger's equation should be normalized with respect to the inner product (3.56) of $L^2_\star(\mathbb{R}^1)$. The original operatorial form of physical states $\psi_{phy}(\hat{x},\hat{t})$ should then be a part of a suitable subspace of $\hat{\mathcal{A}}_\theta$ (3.29), which is different from $\hat{\mathcal{H}}_q$, as for the latter, the norm is derived from the inner product defined for $L^2_\star(\mathbb{R}^2)$ (3.40).

3.3 Forced harmonic oscillator in Moyal space-time

After deliberating on the fundamental formalism, we are now prepared to discuss about one of the possible applications. As previously indicated, the forced harmonic oscillator [99, 100] has many applications in physics, particularly in quantum optics, making it a useful and straightforward model to investigate the QM of time-dependent systems. In order to test this, we set up a forced harmonic oscillator in a NC space-time in 1+1 dimensions to see if non-commutativity has any impact on the geometric phase when the system is evolved adiabatically in a closed loop in the space of parameters over a time period $\mathcal{T}$. These effects, as we shall see, are not present in the system in a classical i.e. commutative space-time setting.
In commutative space-time, the Hamiltonian of a 1 D forced Harmonic oscillator (FHO) is given by,

$$H = \frac{\hat{p}_q^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2 + f(t)\hat{q} + g(t)\hat{p}_q \tag{3.57}$$

where $\hat{p}_q$ and $\hat{q}$ satisfies the usual Heisenberg algebra : $[\hat{q},\hat{p}_q] = i$ and $(f(t)), g(t))$ are a pair of suitably chosen periodic functions of time with time period $\mathcal{T}$ to be specified later.
Now, in the presence of non-commutativity in space-time, we consider the Hamiltonian to be given by the following simple form of operator ordering in order to make it hermitian:

$$\hat{H} = \frac{\hat{P}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2 + \frac{1}{2}[f(\hat{T})\hat{X} + \hat{X}f(\hat{T})] + g(\hat{T})\hat{P}_x \tag{3.58}$$

where $\hat{T}$ and $\hat{X}$ are used in place of the variables in the coordinate space from (3.32). With this Hamiltonian in the Schrödinger constraint equation (3.48) we can take its overlap with the coherent state basis (3.38), to obtain using (3.32),

$$i\partial_t(x,t|\psi_{phy}) = (x,t|\hat{H}|\psi_{phy}) = \left[\frac{P_x^2}{2m} + \frac{1}{2}m\omega^2 X_\theta^2 + \frac{1}{2}\{f(T_\theta)X_\theta + X_\theta f(T_\theta)\} + g(T_\theta)P_x\right]\psi_{phy}(x,t) \quad (3.59)$$

[4] where we have made use of the coherent state representation of the phase space variables, which are given by (3.43,3.44,3.47) [92],

$$P_t = -i\partial_t;\;\; P_x = -i\partial_x;\;\; X_\theta = x + \frac{\theta}{2}(\partial_x - i\partial_t);\;\; T_\theta = t + \frac{\theta}{2}(\partial_t + i\partial_x)$$

Interestingly, these X_θ and T_θ may be connected to commutative x and t, introduced in (3.34) by a similarity transformation as,

$$X_\theta = SxS^{-1};\quad T_\theta = S^\dagger t(S^\dagger)^{-1} \quad (3.60)$$

where

$$S = e^{\frac{\theta}{4}(\partial_t^2+\partial_x^2)}e^{-\frac{i\theta}{2}\partial_t\partial_x} \quad (3.61)$$

is a non-unitary operator that can be used to define a map from $L^2_*(\mathbb{R}^1)$ to $L^2(\mathbb{R}^1)$ as,

$$S^{-1} \,:\, L^2_*(\mathbb{R}^1) \;\rightarrow\; L^2(\mathbb{R}^1)$$
$$S^{-1}\big(\psi_{phy}(x,t)\big) \;:=\; \psi_c(x,t) \;\in\; L^2(\mathbb{R}^1) \quad (3.62)$$

where the elements of the $L^2(\mathbb{R}^1)$ space are indicated by the notation ψ_c. Now it is simple to check that the inner product (3.56) in $L^2_*(\mathbb{R}^1)$ is the same as its commutative equivalent in $L^2(\mathbb{R}^1)$. (see Appendix-C)

$$\Big\langle \psi_{phy}\,,\,\phi_{phy}\Big\rangle_{*,\,t} = \langle\psi_c\,,\phi_c\rangle_t \;\;\forall\,\psi_{phy},\phi_{phy}\,\in\,L^2_*(\mathbb{R}^1) \quad (3.63)$$

where we used integration by parts and dropped many boundary terms. This equivalence demonstrates that, once in $L^2(\mathbb{R}^1)$, we can certainly trade the non-local Voros star product for the local pointwise multiplication that occurs in commutative quantum mechanics. It should, however, be stressed that while the final outcomes of the integrations are same on both sides of (3.63), the integrands themselves are not: $\rho_\theta(x,t) = \psi^*_{phy}(x,t)\star\psi_{phy}(x,t) \neq \psi^*_c(x,t)\psi_c(x,t) = |\psi_c(x,t)|^2$ since certain boundary terms had to be omitted in order to get (3.63). This means that unlike $\rho_\theta(x,t)$, $|\psi_c(x,t)|^2$ can not be regarded as the probability density for witnessing the particle at the position x at time t. As previously demonstrated, ρ_θ (3.54) is the probability density in NC spacetime, but $|\psi_c(x,t)|^2$ can only be read as probability density in the commutative limit $\theta\rightarrow 0$. Additionally, $\psi_{phy}(x,t)$ and $\psi_c(x,t)$ occurring in the integrands themselves are elements belonging to two distinct algebras: commutative and non-commutative, which are not obviously isomorphic to one another [105], and the effect of non-commutativity becomes apparent through the considered dynamical model in a different way [106] when we change from $L^2_*(\mathbb{R}^1)$ to $L^2(\mathbb{R}^1)$ with the aid of the non-unitary transformation S^{-1}.

[4] The hermiticity of the Hamiltonian in the right hand side of (3.59) can be established as already discussed in 3.2.1 after (3.44).

Using (3.60,3.62), (3.59) can now be recast in the following form,

$$i\partial_t\psi_c(x,t) = \left[\frac{p_x^2}{2m} + \frac{1}{2}m\omega^2x^2 + \frac{1}{2}\left\{f(S^{-1}S^\dagger t(S^\dagger)^{-1}S)x + xf(S^{-1}S^\dagger t(S^\dagger)^{-1}S)\right\}\right.$$
$$\left. + g(S^{-1}S^\dagger t(S^\dagger)^{-1}S)p_x\right]\psi_c(x,t) \tag{3.64}$$

Further using the $S^{-1}S^\dagger = e^{i\theta\partial_t\partial_x} := U$, with U being unitary, (3.64) can be simplified as,

$$i\partial_t\psi_c(x,t) = \left[\frac{p_x^2}{2m} + \frac{1}{2}m\omega^2x^2 + \frac{1}{2}\{f(UtU^{-1})x + xf(UtU^{-1})\} + g(UtU^{-1})p_x\right]\psi_c(x,t) \tag{3.65}$$

By applying the Hadamard identity such as $UtU^{-1} = t - \theta p_x$ up to first order in θ assuming that θ is extremely tiny, we now further use the Taylor expansion as

$$F(t-\theta p_x) \approxeq F(t) - \theta p_x\dot{F}(t) \tag{3.66}$$

where $F(t)$ collectively denotes $f(t)$ and $g(t)$, and $\dot{F}(t)$ is their first order temporal derivatives. Remember that $f(t)$ and $g(t)$ are periodic functions of 't', so will be $\dot{f}(t)$ and $\dot{g}(t)$. Now we employ the requirement that $\dot{f}$ and $\dot{g}$ is slowly changing functions of time. In the system under discussion, this will make it easier to utilize an adiabatic approximation. Here, we wish to emphasize that if we were to assume that $f(t)$ and $g(t)$ to be adiabatic in nature, then the second and higher order derivatives of f and g would have to be eliminated from our calculations due to adiabatic approximation, and the geometric phase would disappear in that case. We will thus pick these periodic functions $f(t)$ and $g(t)$ very carefully, ensuring that just their first order derivatives are slowly evolving periodic functions of time.

Finally, employing (3.66) in (3.65), we obtain the theory's effective commutative time-dependent Schrödinger equation as,

$$i\partial_t\psi_c(x,t) = H_c\psi_c(x,t) \tag{3.67}$$

where H_c is the corresponding effective commutative Hamiltonian and is given as

$$H_c = \alpha(t)p_x^2 + \beta x^2 + \gamma(t)(xp_x + p_xx) + f(t)x + g(t)p_x = H_{GHO} + f(t)x + g(t)p_x \tag{3.68}$$

where H_{GHO} stands for Hamiltonian of a generalised time dependent harmonic oscillator and the time dependent and independent coefficients occurring there are given by,

$$\alpha(t) = \frac{1}{2m} - \theta\dot{g}(t);\quad \beta = \frac{1}{2}m\omega^2;\quad \gamma(t) = -\frac{1}{2}\theta\dot{f}(t) \tag{3.69}$$

while the other two terms indicate perturbations that are linear in position and momentum in coordinate basis. We attempt to diagonalize H_{GHO} first as in [108] before attempting to do so for the entire Hamiltonian. And to that end, we introduce the following annihilation and creation operators:

$$a(t) = A(t)[x + (B(t) + iC(t))p_x] \tag{3.70}$$

where,

$$A(t) = \sqrt{\frac{\beta}{\Omega(t)}} \quad \text{where} \quad \Omega(t) = 2\sqrt{\alpha\beta - \gamma^2};$$

$$B(t) = \frac{\gamma(t)}{\beta}; \quad C(t) = \frac{\Omega(t)}{2\beta} \tag{3.71}$$

are time dependent real functions of time. Correspondingly,

$$a^\dagger(t) = A(t)[x + (B(t) - iC(t))p_x] \tag{3.72}$$

fulfilling $[a(t), a^\dagger(t)] = \mathbf{1}$.
As a result of their dependence on α, γ, and therefore on $\dot{f}(t)$ and $\dot{g}(t)$, the coefficients A, B, C and Ω are all time-dependent. As a consequence, we may conclude that A, B, C are slowly changing functions of time and we will disregard the second and higher order time derivatives of these variables in our adiabatic approximation. As a result, the Hamiltonian (3.68) may be expressed using the ladder operator as,

$$H_c = \Omega(t)\Big(a^\dagger(t)a(t) + \frac{1}{2}\Big) + P(t)a(t) + \bar{P}(t)a^\dagger(t) \tag{3.73}$$

where,

$$P(t) = A(t)[C(t)f(t) + i(B(t)f(t) - g(t))] = \sqrt{\frac{\beta}{\Omega(t)}}\left[\frac{\Omega}{2\beta}f(t) + i\left(\frac{\gamma(t)f(t)}{\beta} - g(t)\right)\right]$$

and $\Omega(t)$ can be recognised with the "instantaneous frequency". Although this $\Omega(t)$ may be considered to be a slowly varying function of time, consistent with the adiabatic nature, statuses of the other time dependent functions $P(t)$ and $\bar{P}(t)$ are rather obscure since they include terms like the product of a fast changing and a slowly varying function of time. Therefore, we cannot claim that the whole Hamiltonian evolves slowly under the temporal evolution.
Let's now try to find an appropriate time-dependent unitary transformation $\mathcal{U}(t)$ valued in appropriate Lie group by which the wave function $\psi_c(x,t)$ in (3.67) transforms as

$$\psi_c(x,t) \;\rightarrow\; \tilde{\psi}_c(x,t) = \mathcal{U}(t)\psi_c(x,t); \quad \mathcal{U}^\dagger(t)\mathcal{U}(t) = 1 \tag{3.74}$$

Correspondingly, the Hamiltonian will transform under this time-dependent unitary transformation as

$$H_c \;\rightarrow\; \tilde{H}_c = \mathcal{U}(t)H_c\mathcal{U}^\dagger(t) - i\mathcal{U}(t)\partial_t\mathcal{U}^\dagger(t) \tag{3.75}$$

This reflects that the time evolution of the transformed states $\tilde{\psi}_c(x,t)$ is not governed by just $\mathcal{U}H_c\mathcal{U}^\dagger$ anymore, rather by an effective Hamiltonian $\tilde{H}_c$, which is obtained by augmenting it by a suitable "connection" like term to $\mathcal{U}H_c\mathcal{U}^\dagger$:

$$i\partial_t\tilde{\psi}_c(x,t) = \tilde{H}_c\tilde{\psi}_c(x,t) \tag{3.76}$$

The instantaneous energy eigenstates of the Hamiltonian $\tilde{H}_c$ are no longer the same as those of H_c. We now want to exclude the terms in $\tilde{H}_c$ that include fast variables, such as $P(t)$ and $\bar{P}(t)$, so that we are only left with the first term in (3.73), which may be achieved by making an appropriate choice for

$\mathcal{U}(t)$. In fact, such a $\mathcal{U}(t)$ can be easily constructed as,

$$\mathcal{U}(t) = e^{-(wa - \bar{w}a^{\dagger} + il)}, \quad w \in \mathbb{C},\ l \in \mathbb{R} \tag{3.77}$$

where w and l are given by,

$$\begin{aligned} w(t) &= -ie^{i\int_0^t dt'\,\Omega(t')} \int_0^t dt''\, P(t'')e^{-i\int_0^{t''} d\tilde{t}\,\Omega(\tilde{t})} \\ l(t) &= \int_0^t dt' \left[\Omega(t')|w(t')|^2 - \frac{i}{2}\Big(w(t')\dot{\bar{w}}(t') - \dot{w}(t')\bar{w}(t')\Big)\right] \end{aligned} \tag{3.78}$$

This structure of $\mathcal{U}(t)$ in (3.77) may be related to the non-abelian Heisenberg group, which can be constructed by exponentiating the components of the Heisenberg Lie algebra $\mathfrak{h}$, which is generated by $(a, a^{\dagger}, \mathbb{I})$, where $\mathbb{I}$ is the central extension. Using this, (3.75) becomes,

$$\tilde{H}_c = \Omega(t)(a^{\dagger}(t)a(t) + \frac{1}{2}) = H_{GHO} \tag{3.79}$$

Therefore, by providing a time-dependent unitary transformation (3.77), we can demonstrate how the entire Hamiltonian H_c (3.68) transforms into a generalized harmonic oscillator Hamiltonian (3.79). Since $\tilde{H}_c$ has been diagonalized, the computation of the geometric phase gained by $\psi_c(x,t)$ is simplified to that of determining the equivalent phase acquired by $\tilde{\psi}_c(x,t)$, which we address in the next section.

3.3.1 Evolution of the ladder operators and appearance of geometric phase:

By employing the diagonalized form of the corresponding Hamiltonian $\tilde{H}_c$ (3.79) within our adiabatic approximation, the time-evolved expression of $\tilde{\psi}_c(x,t)$ in (3.76) becomes significantly simpler to compute compared to its precursor wave function $\psi_c(x,t)$ in (3.67), which evolves with the Hamiltonian H_c. In fact, as demonstrated in [92], it becomes much more straightforward to isolate the geometric phase within the Heisenberg picture. In this section, we undertake this task.
The Heisenberg equation of motion using (3.79) for the operators $a(t)$ and $a^{\dagger}(t)$ is given by,

$$\frac{da^{\dagger}}{dt} = \xi(t)a^{\dagger} + X(t)a \qquad \text{and} \qquad \frac{da}{dt} = \bar{\xi}(t)a + \bar{X}(t)a^{\dagger} \tag{3.80}$$

where

$$X(t) = -A^2(\dot{C} + i\dot{B}); \quad \xi(t) = Y(t) + i\Omega(t); \quad Y(t) = A[(2\dot{A}C + A\dot{C}) + iA\dot{B}] \tag{3.81}$$

To solve the first order differential equation of the ladder operators given by (3.80), we can decouple the equations in (3.80) , by taking their time derivatives and combining them suitably to get [71],

$$\frac{d^2a^{\dagger}}{dt^2} + Z_1(t)\frac{da^{\dagger}}{dt} + Z_2(t)a^{\dagger} = 0; \qquad \frac{d^2a}{dt^2} + \bar{Z}_1(t)\frac{da}{dt} + \bar{Z}_2(t)a = 0 \tag{3.82}$$

where

$$Z_1(t) = -\left(\frac{\dot{X}}{X} + \xi + \bar{\xi}\right); \qquad Z_2(t) = -(\dot{\xi} - \frac{\dot{X}}{X}\xi + X\bar{X} - \xi\bar{\xi}) \tag{3.83}$$

where the time-dependent variables $X(t), \xi(t)$, are defined in (3.81). So far, all of the expressions obtained here are exact. The adiabaticity of $\alpha(t)$ and $\gamma(t)$ in (3.69) will now be implemented. It follows that $\alpha(t), \gamma(t)$ (3.69) also changes adiabatically with time as we have assumed that $\dot{f}$ and $\dot{g}$ are slowly varying periodic functions of time. A, B and C follow the same order of adiabaticity since they depend linearly on α, β and γ, respectively (3.71). In this case, $\dot{F} \sim \epsilon$, and, $\ddot{F} \sim \epsilon^2$, where F stands for A, B, C collectively. Eventually, while taking adiabatic time evolution in this regard, we will dispose of second and higher order time derivatives ($\sim \epsilon^2$). To continue, we write (3.82) in its standard form. To do so, we introduce another time dependent operator, $b(t)$ as,

$$a^\dagger(t) = b^\dagger(t) e^{-\frac{1}{2}\int_0^t Z_1(\tau)d\tau} \tag{3.84}$$

With this the first equation of (3.82) can be recast in terms of $b^\dagger$

$$\frac{d^2 b^\dagger(t)}{dt^2} + \tilde{Z}_2(t) b^\dagger(t) = 0 \tag{3.85}$$

where,

$$\tilde{Z}_2 = Z_2 - \frac{1}{2}\dot{Z}_1 - \frac{Z_1^2}{4} \approx \Omega^2 - \Omega\tilde{W} + i(2\Omega\frac{\dot{A}}{A} + W) \sim \mathcal{O}(\epsilon) = U + iV(say) \tag{3.86}$$

where we have taken $\frac{\ddot{C}+i\ddot{B}}{\dot{C}+i\dot{B}} = W + i\tilde{W} \sim \mathcal{O}(\epsilon)$. Then (3.85) can be re-written in the following form:

$$\frac{d^2 b^\dagger(t)}{dt^2} + (U + iV) b^\dagger(t) = 0 \tag{3.87}$$

As we are operating in the adiabatic regime, it is evident from (3.86) that both U and V vary gradually over time. Then, using the WKB approximation for complex potential [107] formula, we get,

$$b^\dagger(t) = b^\dagger(0)\left[\frac{C_1}{\sqrt{|\chi(t)|}} exp\left(\int_0^t (i\chi(\tau) - \phi(\tau))d\tau\right) + \frac{C_2}{\sqrt{|\chi(t)|}} exp\left(\int_0^t (-i\chi(\tau) + \phi(\tau))d\tau\right)\right] \tag{3.88}$$

where $\sqrt{U + iV} = \chi + i\phi$ so that (3.88) can be applied to find the general solution of the differential equation (3.87). In our case, χ and ϕ are given as,

$$\chi = \sqrt{\frac{\sqrt{U^2 + V^2} + U}{2}} \approx \sqrt{U + \frac{V^2}{4U}} \approx \sqrt{U} \approx \Omega - \frac{\tilde{W}}{2}$$

$$\phi = \sqrt{\frac{\sqrt{U^2 + V^2} - U}{2}} \approx \sqrt{\frac{V^2}{4U}} \approx \frac{\dot{A}}{A} + \frac{W}{2}$$

where all terms of second and higher order derivatives are disregarded. Keeping in mind that only the first term of the exponential in (3.88) produces the appropriate dynamical phase for $a^\dagger$, we are going to set $C_2 = 0$. Taking into account every expression we obtain, at $t = \mathcal{T}$,

$$b^\dagger(\mathcal{T}) \approx b^\dagger(0) exp\left(\int_0^{\mathcal{T}} [i\Omega - \frac{d}{d\tau}(lnA) - \frac{1}{2}(W + i\tilde{W})]d\tau\right) \tag{3.89}$$

Given the periodic boundary condition is $A(T) = A(0)$ and that the integrands are exact differentials, we may eliminate the following terms from the exponent:

$$\int_0^{\mathcal{T}} \Big(\frac{\dot{A}}{A} + \frac{W + i\tilde{W}}{2}\Big) d\tau = \int_0^{\mathcal{T}} \frac{1}{2}\frac{d}{d\tau}\Big(lnX\Big) d\tau = 0$$

where X is given in (3.81). Substituting (3.89) in (3.84), we get

$$a^{\dagger}(\mathcal{T}) = b^{\dagger}(\mathcal{T})\ exp\ \Big(\int_0^{\mathcal{T}} \Big[\frac{d}{d\tau}(ln\, X + A^2 C) + iA^2 B\Big] d\tau\Big) = b^{\dagger}(\mathcal{T}) e^{i\int_0^{\mathcal{T}} A^2 \dot{B} d\tau} \tag{3.90}$$

The expression of A, B, C are given in (3.71). Finally making use of (3.89) in (3.90) we can ultimately write,

$$\begin{aligned} a^{\dagger}(\mathcal{T}) &= a^{\dagger}(0) exp\left(i\int_0^{\mathcal{T}} \Omega\ d\tau\right) exp\left(i\int_0^{\mathcal{T}} A^2 \dot{B}\ d\tau\right) \\ &= a^{\dagger}(0) exp\left[i\int_0^{\mathcal{T}} \Omega\ d\tau + i\int_0^{\mathcal{T}} \left(\frac{1}{\Omega}\right)\frac{d\gamma}{d\tau}\ d\tau\ \right] \end{aligned} \tag{3.91}$$

The two terms in the exponents clearly represent the dynamical and geometric phase respectively. As a result of the Hamiltonian's adiabatic evolution around a closed loop Γ in the parameter space in time $\mathcal{T}$, the ladder operator acquired an extra phase over and above the dynamical phase $e^{i\int \Omega(t)dt}$, is represented by the second term in the exponential of (3.91). This geometric phase, often referred to as the Berry phase , may be expressed in a more recognizable manner by expressing it as a functional of Γ as

$$\Phi_G[\Gamma] = \oint_{\Gamma=\partial S} \frac{1}{\Omega}\nabla_{\mathbf{R}}\gamma . d\mathbf{R} = -\frac{\theta}{2}\int\int_S \nabla_{\mathbf{R}}\left(\frac{1}{\Omega}\right) \times \nabla_{\mathbf{R}}\Big(\dot{f}(t)\Big) . d\mathbf{S} \tag{3.92}$$

where we have used $\frac{d}{d\tau} = \frac{d\mathbf{R}}{d\tau} . \nabla_{\mathbf{R}}$, $\mathbf{R} \equiv (\alpha, \beta, \gamma)$ being a vector in the parameter space whose components are time dependent. We used Stoke's theorem in the second equality to change the line integral into the surface integral. Clearly, this geometrical phase vanishes in the commutative limit $\theta \to 0$. It should be noted in this context that that by switching over to the Schrödinger picture, we can relate the geometric phase shift achieved in the Heisenberg picture to a more recognisable type of Berry phase acquired by the state vectors.
It is worth noting that had we worked in the Schrödinger's picture, the primary wave function $\psi_c(x,t)$ (3.74) would have acquired the same geometric phase $\Phi_G[\Gamma]$ over the time interval $\mathcal{T}$. This holds true because the original state $\psi_c(x,t)$ can be obtained by applying the linear and unitary operator $\mathcal{U}^{\dagger}(t)$ on $\tilde{\psi}_c$ as $\mathcal{U}^{\dagger}(t)\tilde{\psi}_c(x,t)$, where $\mathcal{U}(t)$ is given by (3.77,3.78). Since the Berry phase is a constant number (3.92) for a given Γ, it remains unaffected by the action of the Heisenberg group element. Before concluding this section, we would like to highlight some important observations that we can make:

- From (3.92), it is clear that the geometric phase relies on the parameter γ from (3.69), which disappears when $\theta = 0$. Therefore, the non-commutativity of space-time is necessary for the emergence of the geometric phase in this context. In this respect, it should be noted that the authors in [108] have demonstrated that a system must contain the dialatation term in the Hamiltonian in order to generate the geometric phase. The forced harmonic oscillator (3.57), our initial system, did not have such a term, but by putting the system in an NC space-time, we were able to obtain such a term in the effective commutative Hamiltonian (3.68).

- The Hamiltonian of the F.H.O in (3.68) can be expanded in terms of the generators of the double photon algebra [109]. For that let us define

$$K_+ = \frac{ix^2}{2}, \;\; K_- = \frac{ip^2}{2}, \;\; K_0 = \frac{i(xp+px)}{4}, \;\; A_+ = x, A_- = p \tag{3.93}$$

Taking $\mathbb{I}$ as a central extension we can show that the above generators satisfy the following Lie algebra:

$$[K_0, K_\pm] = \pm K_\pm, \;\; [K_+, K_-] = -2K_0 \tag{3.94}$$

$$[K_+, A_-] = -A_+, \;\; [K_-, A_+] = A_-, \;\; [K_0, A_+] = \frac{A_+}{2}, \;\; [K_0, A_-] = -\frac{A_-}{2}, \;\; [A_+, A_-] = i\mathbb{I} \tag{3.95}$$

where the sub-algebra (3.94) is the $\mathfrak{su}(1,1)$ Lie-algebra and (3.94),(3.95) as a whole represents the double photon algebra [109].
Now the Hamiltonian (3.68) can therefore be written in terms of the generators (3.93) as

$$H_c = B^\mu(t)K_\mu + f(t)A_+ + g(t)A_-; \;\;\; \mu = +,-,0; \;\;\; B^+ = -2i\alpha(t), B^- = -2i\beta(t), B^0 = -2i\gamma(t) \tag{3.96}$$

The corresponding transformed Hamiltonian $\tilde{H}_c$, on the other hand, takes its value only in the $\mathfrak{su}(1,1)$ sub-algebra.

$$H_c \rightarrow \tilde{H}_c = \mathcal{U}H_c\mathcal{U}^\dagger - i\mathcal{U}\frac{\partial \mathcal{U}^\dagger}{\partial t} = B^\mu(t)K_\mu \tag{3.97}$$

At this point, it is important to note that the Hamiltonians in [108] were analogous to ours and were valued in $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ Lie algebras, for example. Thus, just like in their case, we may also expect the occurrence of the geometric phase in our situation.

3.4 Chapter summary

It is indeed a tough undertaking to identify the footprints of Planck scale phenomena involving NC space-time in a non-relativistic (NR) system. However, there exists literature [110, 111] where authors have attempted to identify feasible scenarios that may aid in determining the impacts induced by quantum structure of space-time in the low energy regime. That inspired us to investigate the effects of Planck scale physics in NR quantum systems, which was the core goal of the current chapter. We summarize our results here very briefly.
We demonstrated that a forced harmonic oscillator system living in a NC space-time develops a geometric phase shift when transported adiabatically along a closed loop in the parameter space Γ. The Hamiltonian of the effective commutative system is demonstrated to be that of a time dependent generalized harmonic oscillator with linear perturbation in momentum and position. The system is then subjected to a appropriate time-dependent unitary transformation to remove the perturbative terms and is then diagonalized. We subsequently determined the equation of motion of the ladder operators in the Heisenberg picture, which reveals that, depending on the geometry of the parameter space, an additional phase emerges when the system is transported adiabatically through a closed loop Γ. This phase, which is computed up to first order in the non-commutative parameter θ in our formula and vanishes in the commutative limit $\theta \rightarrow 0$, is found to be a functional of Γ. In this respect, it should be emphasized that our computation of the geometric phase, carried out in the Heisenberg

picture, as in [71], is a novel method. It can be demonstrated that the geometric phase obtained using this method equals the Hannay's angle, which is produced by the comparable classical adiabatic evolution.

Finally, we would like to discuss a few potential future directions for our work. The authors in [112] had demonstrated an intriguing connection between the energy-time uncertainty relation and the Fubini-Study metric, developed in the projective Hilbert space of a time-dependent quantum system. Later, authors in [113] have noted that coherent states offer a natural framework for realizing the aforementioned relationship and have explicitly shown this in their computations using, respectively, a spin system in a magnetic field and a generalized harmonic oscillator system. In light of [113], it appears that we may extend our investigation to calculate the energy-time uncertainty relation in our system as well by utilizing the Fubini-Study metric of the projective Hilbert space.
Furthermore, it was demonstrated in [99] that a coherent state continues to remain coherent at all times if the system Hamiltonian, with which the operators are evolved (in the Heisenberg picture), is that of a time-dependent forced harmonic oscillator. In presence of space-time noncommutativity, we may extend this computation for our system. For the generalised forced harmonic oscillator (GFHO)- the effective commutative system will generate states that are squeezed coherent states of a typical FHO [114]. This motivates us to investigate the fate of these squeezed coherent states, when evolved over time and whether they have any implications for quantum optics and quantum information theory.
Also notice that space-time noncommutativity can also be incorporated in non-relativistic second quantized formalism. This would be a good launching pad to generalize NCQM, viewed as 0+1 dimensional QFT, to relativistic QFT in NC space-time. This is because, unlike in QM, space and time can naturally be treated on equal footing here as the spatial coordinates do not correspond to any observables anymore and rather corresponds to just c-numbered parameters (labelling the continuous spatial degrees of freedom of the system). Importantly, the second quantized formalism allows us to explore the influence of space-time noncommutativity on the one-particle sector of quantum field theory for a generic external potential [31], which is analogous to the first quantized formalism's NC quantum mechanical analysis. These are among some of the issues which can possibly be investigated in future.

Chapter 4
Relativistic particle in κ-Minkowski space-time

Having discussed the effects of NC space-time in a non-relativistic dynamical model, we now move on to discuss about the impacts of quantum space-time with Lie algebraic type of noncommutativity, namely κ-Minkowski noncommutativity on a relativistic system. Employing the deformed Poincare symmetries of such space-time at classical level, we try to develop the Lagrangian describing the dynamics of a single relativistic massive spinless free particle. This uncovers a plausible regime of QG where we see that QG effects are present only through a mass-scale $\kappa = \sqrt{\frac{\hbar}{G}}$, in the case where both $\hbar, G \to 0$ but holding their ratio fixed.
Interestingly, a recent proposal [50] suggests that the quantum origin of gravity could have an impact even in the regime of weak Newtonian gravity through quantum superposition and entanglement in the infrared region [51], which has connections to quantum gravity-induced entanglement of masses (QGEM). This motivates the search for systems that exhibit some robust features which may survive in a chosen regime of a future theory of quantum gravity [115]. To explore such a scenario, we consider a system where the associated length scale l is much larger than the Planck length ($l >> l_P$), but the mass scale m is comparable to κ ($m \leq \kappa = \sqrt{\frac{\hbar}{G}}$). In this case, we effectively examine a situation where both $\hbar$ and G tend to zero as discussed above while their ratio is held fixed at κ. The operator valued NC coordinate algebra, is by default, characterized by NC parameters with length dimensions that can be identified with Planck length scale $\sim l_P$. Such noncommutativity naturally arises in Lie-algebra-type deformations, serving as the deformation parameter, which effectively reduces to κ^{-1} in the corresponding Poisson (or Dirac) bracket of the classical description. To study this, we can consider the well-known κ-Minkowski spacetime originally introduced in [47,53–57], and then proposed again by [58] in the context of double-special relativity [116,117] aiming to extend the special theory of relativity (STR) by incorporating yet another observer-independent scale κ. As we shall see that the presence of the mass scale κ alone in the classical analysis can have a significant impact on the system dynamics, curving the energy-momentum or simply the momentum space itself [52], thereby deforming the dispersion relation even for a single particle. This curvature of momentum space may lead to the abandonment of absolute locality in favor of a concept called "relative locality" [52].
Interestingly, Max Born speculated about the necessity of curved momentum space in the context of quantum gravity as early as 1938 [48]. Later in the year 2000, it was shown again in [46,47,49] that the curved momentum space can be thought of as Hopf dual to NC space-time and the term 'co-gravity' was coined by the authors in context of curved momentum space. This resembles the well-known curvature of 3-velocity space in STR, where the deformation parameter is the speed of light c. The manifestation of curved 3-velocity space is observed through the non-linear addition of velocities,

which is generically noncommutative (NC) and non-associative in nature, unless they are co-linear, where the 3-velocity space becomes flat in the $c \to \infty$ limit,. Similar to STR, the flat limit in the curved momentum space is only recovered as $\kappa \to \infty$. A concrete realization of curved momentum space was also provided in a $2+1$-dimensional system, where a spin-less relativistic point particle interacts with Einstein's gravity. In this case, Einstein's gravity is a topological theory and can be formulated as a non-abelian ($ISO(2,1)$) Chern-Simons theory [118]. Although Einstein's theory is no longer topological in the realistic $3+1$-dimensional spacetime, there is indirect evidence suggesting a similar behavior [119].

In this chapter, we study the symmetry of the κ-Minkowski space-time. We find that it becomes essential to deform the actions of the Poincare generators on the coordinate operators, so that they are now allowed to take values in the universal enveloping algebra, obtained from the undeformed Poincare generators, but in a manner that the Poincare algebra-by itself undergoes no deformation. In this context, we find it convenient to follow the template of [120], where the symmetries of Moyal space-time has been discussed. Also, we could not adopt the twisted Hopf-algebraic framework, as no twist or star product seems to exist in this case, although there exist other variants of κ-Minkowski spacetime [57, 59, 121–124], where both the star product/twist can be readily introduced.

We then carry out the dynamical analysis entirely at the classical level by demoting all the generators to commuting classical variables, where the various commutator brackets now correspond to usual symplectic brackets $\{.,.\}$. We then show how these brackets can, in turn, be interpreted as the Dirac bracket of a first order constrained system, describing the dynamics of the relativistic particles moving in the κ-Minkowski spacetime. We find that $\{x^i, p_j\}$ undergoes a momentum dependent deformation, so that the momentum space can now be identified with curved space, which, however, is not quite Riemannian. Nevertheless, an invariant line element of the form $ds = \sqrt{g_{\mu\nu}dp^\mu dp^\nu}$ can be introduced, so that the geodesic distance in the momentum space can easily be computed, enabling us to obtain the deformed dispersion relation which eventually helps us to identify renormalised observable mass M. And for any isolated fundamental particle the mass scale κ can serve as an upper bound for this renormalized mass : $M < \kappa$ for a certain choice of the NC parameters. Interestingly, it turns out that this κ is the only surviving natural mass scale in this classical system.

The chapter is organized as follows: In section-4.1, we revisit the deformed symmetries of κ-Minkowski space-time using the formulation [120], followed by a discussion on the deformed co-products of Poincare generators and a Heisenberg-double construction in Hopf-algebroid framework for a consistency check of the phase-space algebra coming from the deformed co-product of the momentum. In section-4.2 we construct the first order Lagrangian of a free massive relativistic spinless particle moving in κ space-time which obeys the same symmetry as that of the space-time. In section-4.2.1 we explicitly derive the deformed mass-shell condition using the computation of geodesic distance in momentum space. In section-4.2.2, we have shown the non-canonical transformation between noncommutative and commutative coordinates, which helps us to find out the explicit form of the deformed Lorentz generators. Later in section-4.2.3 we show the invariance of the Lagrangian under the deformed symmetries and derived the Poincare generators using Noether's prescription which corroborates with the ones derived in the preceding sub-section. In section-4.2.4 we have studied the feasibility of lifting the infinitesimal Lorentz symmetry to a finite one on the NC coordinate, using the non-canonical transformation between commutative and NC coordinates derived earlier in section-4.2.2 and found an explicit Lorentz invariant 'interval' under the finite transformation which is, however, a function of phase-space variables. Finally, in section-4.3 we conclude with some overview and future directions.

4.1 Deformed symmetries of κ-Minkowski space-time

The algebra of κ Minkowski space-time $\hat{\mathcal{M}}$ is given as

$$[\hat{X}^\mu, \hat{X}^\nu] = i\hat{\theta}^{\mu\nu} = i(a^\mu \hat{X}^\nu - a^\nu \hat{X}^\mu) \quad (4.1)$$

where $\hat{X}^\mu$ are the operator valued NC coordinates and a^μ is a set of four real scalar constants that may be associated with the set of four deformation parameters. These deformation parameters have the order of κ^{-1} : $a^\mu \sim \kappa^{-1}$. Despite its appearance, a^μ doesn't behave as a vector under Lorentz transformation; instead, all of its components persist to be invariant scalars in every Lorentz frame.[1] It is simple to demonstrate that under normal infinitesimal Poincare transformation of coordinates provided by

$$\delta\hat{X}^\mu = \epsilon^\mu, \qquad \delta\hat{X}^\mu = \omega^\mu{}_\alpha \hat{X}^\alpha \quad (4.2)$$

the coordinate algebra (4.1) is not invariant (where ϵ^μ and $\omega^{\mu\nu} = -\omega^{\nu\mu}$ with $|\epsilon^\mu|, |\omega^{\mu\nu}| << 1$ are infinitesimal parameters corresponding to translations and homogeneous Lorentz transformations). This suggests that, some form of modified Poincare transformation is necessary to preserve the κ-Minkowski algebra (4.1). We will derive the deformed symmetries associated with Poincare transformation in this section. For that we won't deform the $\mathfrak{iso}(1{,}3)$ Lie algebra between the generators of Poincare transformation, rather their actions on the operator valued coordinates will be deformed. This guarantees that the translation generators will transform as usual i.e. like a 4-vector under Lorentz transformation[2]. Despite the fact that this has been demonstrated by a number of authors in the literature [122, 123, 125–128], we revisit the issue in order to provide a systematic derivation for obtaining the deformed transformations, using the model provided by F. Koch *et.al.* [120]. As we'll see, this task is crucial for our discussion in the parts that follow, where we shall develop a relativistic action for a spin-less free massive particle in κ-Minkowski space-time.
Now the Lie algebra between the Poincare generators, $\mathfrak{iso}(1{,}3)$, is given by,

$$\begin{aligned}
[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}] &= i(\eta_{\nu\rho}\hat{M}_{\mu\sigma} + \eta_{\mu\sigma}\hat{M}_{\nu\rho} - \eta_{\mu\rho}\hat{M}_{\nu\sigma} - \eta_{\nu\sigma}\hat{M}_{\mu\rho}) \\
[\hat{M}_{\mu\nu}, \hat{P}_\rho] &= i(\eta_{\nu\rho}\hat{P}_\mu - \eta_{\mu\rho}\hat{P}_\nu) \\
[\hat{P}_\mu, \hat{P}_\nu] &= 0
\end{aligned} \quad (4.3)$$

Here $\hat{M}_{\mu\nu}$ and $\hat{P}_\mu$ denote Lorentz and translation generators respectively. As was previously indicated, we want to identify deformed transformations in the coordinate operator without distorting the Lie algebra (4.3) among the Poincare generators. This makes it abundantly evident that $\hat{P}$'s transformation under the Poincare generators remains unaltered, giving us its usual infinitesimal transfor-

[1] This means that a^μ's commute with all the Poincare generators : $[M_{\nu\lambda}, a^\mu] = [P_\nu, a^\mu] = 0$. However, we can still introduce $a_0 = a^0$ and $a_i = -a^i (i = 1, 2, 3)$ as another set of scalars. Here too, we can raise/lower indices using $\eta_{\mu\nu}$ and formally write $a_\mu = \eta_{\mu\nu} a^\nu$ and we can construct $a^\mu a_\mu = a_0^2 - \vec{a}^2$. Further depending on whether $a^\mu a_\mu > 0, < 0, = 0$ we can refer to it as time-like, space-like and null respectively [125].

[2] One can even deform the $\mathfrak{iso}(1{,}3)$ Lie algebra itself to obtain κ-Poincare algebra, which too can generate deformed transformation of the coordinates, keeping the algebra (4.1) stable under such transformation [125]. However in that case, the translation generator will not transform as vectors under Lorentz transformation; instead they will follow enveloping algebra valued transformation.

mation that corresponds to translation and the Lorentz transformation as

$$\delta \hat{P}_\mu = 0 \qquad \text{and} \quad \delta \hat{P}_\mu = \omega_\mu{}^\alpha \hat{P}_\alpha \tag{4.4}$$

respectively.
We can anticipate, at this stage that we must deform the infinitesimal transformations (4.2) and thus deform the commutators $[\hat{M}_{\mu\nu}, \hat{X}_\rho]$ as well as $[\hat{P}_\mu, \hat{X}_\nu]$ in order to maintain consistency of the coordinate algebra (4.1) under the influence of the Poincare generators while satisfying the Lie algebra (4.3) above. In light of this, let's propose the following ansatz for the deformed brackets:

$$[\hat{M}_{\mu\nu}, \hat{X}_\rho] = i(\eta_{\mu\rho}\hat{X}_\nu - \eta_{\nu\rho}\hat{X}_\mu) + i\psi_{\mu\nu\rho}(\hat{P}, \hat{M}) \tag{4.5}$$

$$[\hat{P}_\mu, \hat{X}_\nu] = -i\eta_{\mu\nu}\phi(\hat{P}) + i\chi_{\mu\nu}(\hat{P}, \hat{M}) \tag{4.6}$$

Note that the deformations are entirely contained in $\psi_{\mu\nu\rho}, \phi, \chi_{\mu\nu}$ which we would like to depend entirely on the NC parameter a_μ linearly and also be Lie algebra valued i.e. depend on $\hat{M}$ and $\hat{P}$ linearly, rather than being valued in the universal enveloping algebra, so as to enable us to look for a simple and almost unique solution [125]. However, as it turns out this demand can't be complied with entirely and deformations involving higher orders of a_μ and valued in universal enveloping Lie algebra, must be invoked in a special case, particularly in the deformed Heisenberg algebra (4.6), in order to get consistent solutions (This point will be discussed thoroughly later in this section). In order for the brackets (4.5,4.6) to provide the expected results in the commutative limit, we additionally require that $\psi_{\mu\nu\rho}(\hat{P}, \hat{M}; a) \to 0, \phi(\hat{P}; a) \to 1$, and $\chi_{\mu\nu}(\hat{P}, \hat{M}; a) \to 0$ for $a_\mu \to 0$. Let's also note that the deformations ψ and ϕ, χ have a specific dependency on the generators $\hat{P}$ or $\hat{M}$, and that dependence must be taken into account so that the dimensions of the deformations match the left hand sides of (4.5) and (4.6) respectively.

We utilize two different cases of consistency conditions [129] to solve $\psi_{\mu\nu\rho}(\hat{P}, \hat{M}; a), \phi(\hat{P}; a)$ and $\chi_{\mu\nu}(\hat{P}, \hat{M}; a)$:

Condition-1: Let us consider the infinitesimal Poincare transformation:

$$\hat{X}^\mu \to \hat{X}'^\mu = \hat{X}^\mu + \delta_\epsilon \hat{X}^\mu, \tag{4.7}$$

with $\delta_\epsilon \hat{X}^\mu = i\epsilon^i[\hat{G}_i, \hat{X}^\mu]$, where $\hat{G}_i \in \{\hat{M}, \hat{P}\}$ are the generators for respective infinitesimal transformation. The Jacobi identities between $\hat{G}_i$ and two NC space time coordinates resulting from the covariance of the NC relations (4.1) under such space-time transformation:

$$[[\hat{X}_\mu, \hat{X}_\nu], \hat{G}_i] + [[\hat{X}_\nu, \hat{G}_i], \hat{X}_\mu] + [[\hat{G}_i, \hat{X}_\mu], \hat{X}_\nu] = 0 \tag{4.8}$$

Condition (4.8) reduces to the following relations after using various forms of $\hat{G}_i$,

$$\begin{aligned} [[\hat{P}_\lambda, \hat{X}_\mu], \hat{X}_\nu] + [[\hat{X}_\mu, \hat{X}_\nu], \hat{P}_\lambda] + [[\hat{X}_\nu, \hat{P}_\lambda], \hat{X}_\mu] = 0 \\ [[\hat{M}_{\rho\sigma}, \hat{X}_\mu], \hat{X}_\nu] + [[\hat{X}_\mu, \hat{X}_\nu], \hat{M}_{\rho\sigma}] + [[\hat{X}_\nu, \hat{M}_{\rho\sigma}], \hat{X}_\mu] = 0 \end{aligned} \tag{4.9}$$

We obtain a set of relations between ψ, χ and ϕ by further incorporating the commutators from (4.1, 4.5, 4.6), given by,

$$i\Big\{\eta_{\lambda\nu}[\phi(\hat{P}),\hat{X}_\mu]-\eta_{\lambda\mu}[\phi(\hat{P}),\hat{X}_\nu]\Big\}+i\Big\{[\chi_{\lambda\mu},\hat{X}_\nu]-[\chi_{\lambda\nu},\hat{X}_\mu]\Big\}+\Big(a_\nu\eta_{\lambda\mu}-a_\mu\eta_{\lambda\nu}\Big)\phi(\hat{P})+\Big(a_\mu\chi_{\lambda\nu}-a_\nu\chi_{\lambda\mu}\Big)=0 \tag{4.10}$$

$$i\Big\{[\psi_{\rho\sigma\mu},\hat{X}_\nu]-[\psi_{\rho\sigma\nu},\hat{X}_\mu]\Big\}+\Big(a_\mu\psi_{\rho\sigma\nu}-a_\nu\psi_{\rho\sigma\mu}\Big)+\Big(\eta_{\rho\mu}a_\sigma-\eta_{\sigma\mu}a_\rho\Big)\hat{X}_\nu+\Big(\eta_{\sigma\nu}a_\rho-\eta_{\rho\nu}a_\sigma\Big)\hat{X}_\mu=0 \tag{4.11}$$

Condition-2:

The Jacobi identities between two $\mathfrak{iso}(3,1)$ generators $\hat{G}_i=\{\hat{M}_{\mu\nu},\hat{P}_\lambda\}$ and coordinate $\hat{X}_\mu$ are provided by the covariance of the relationships presented in (4.5,4.6):

$$[[\hat{G}_i,\hat{G}_j],\hat{X}_\mu]+[[\hat{G}_j,\hat{X}_\mu],\hat{G}_i]+[[\hat{X}_\mu,\hat{G}_i],\hat{G}_j]=0 \tag{4.12}$$

Putting various combinations of Poincare generators and coordinate in (4.12) we get the following conditions:

$$\begin{aligned}
[[\hat{P}_\mu,\hat{P}_\nu],\hat{X}_\lambda]+[[\hat{P}_\nu,\hat{X}_\lambda],\hat{P}_\mu]+[[\hat{X}_\lambda,\hat{P}_\mu],\hat{P}_\nu]&=0\\
[[\hat{M}_{\mu\nu},\hat{M}_{\sigma\rho}],\hat{X}_\lambda]+[[\hat{M}_{\sigma\rho},\hat{X}_\lambda],\hat{M}_{\mu\nu}]+[[\hat{X}_\lambda,\hat{M}_{\mu\nu}],\hat{M}_{\sigma\rho}]&=0\\
[[\hat{M}_{\mu\nu},\hat{P}_\sigma],\hat{X}_\lambda]+[[\hat{P}_\sigma,\hat{X}_\lambda],\hat{M}_{\mu\nu}]+[[\hat{X}_\lambda,\hat{M}_{\mu\nu}],\hat{P}_\sigma]&=0
\end{aligned} \tag{4.13}$$

Now putting the relations (4.3, 4.5, 4.6) in the Jacobi identities (4.13), we get following relations between the deformation functions ψ, ϕ and χ:

$$[\chi_{\nu\lambda}(\hat{P},\hat{M}),\hat{P}_\mu]-[\chi_{\mu\lambda}(\hat{P},\hat{M}),\hat{P}_\nu]=0 \tag{4.14}$$

$$\begin{aligned}
i[\psi_{\sigma\rho\lambda},\hat{M}_{\mu\nu}]-i[\psi_{\mu\nu\lambda},\hat{M}_{\sigma\rho}]+\eta_{\mu\lambda}\psi_{\sigma\rho\nu}+\eta_{\rho\lambda}\psi_{\mu\nu\sigma}+\eta_{\mu\sigma}\psi_{\nu\rho\lambda}+\eta_{\nu\rho}\psi_{\mu\sigma\lambda}&\\
-\eta_{\nu\sigma}\psi_{\mu\rho\lambda}-\eta_{\mu\rho}\psi_{\nu\sigma\lambda}-\eta_{\nu\lambda}\psi_{\sigma\rho\mu}-\eta_{\sigma\lambda}\psi_{\mu\nu\rho}&=0
\end{aligned} \tag{4.15}$$

$$-i\eta_{\sigma\lambda}[\phi(\hat{P}),\hat{M}_{\mu\nu}]+i[\chi_{\sigma\lambda},\hat{M}_{\mu\nu}]-i[\psi_{\mu\nu\lambda},\hat{P}_\sigma]+(\eta_{\mu\sigma}\chi_{\nu\lambda}-\eta_{\nu\sigma}\chi_{\mu\lambda}+\eta_{\mu\lambda}\chi_{\sigma\nu}-\eta_{\nu\lambda}\chi_{\sigma\mu})=0 \tag{4.16}$$

We now construct a suitable (as mentioned above condition-1) ansatz for $\chi_{\mu\nu}(\hat{M},\hat{P}), \phi(\hat{P})$ and $\psi_{\mu\nu\lambda}(\hat{M},\hat{P})$ and insert them back into the equations (4.10,4.11,4.14,4.15,4.16) to solve them explicitly. Note that $\psi_{\mu\nu\lambda}(\hat{M},\hat{P})$ has dimension of length and is antisymmetric in the index μ,ν according to (4.5). As a result, we make the following ansatz of ψ, which is also first order in the NC parameter a_μ, as shown in the following,

$$\psi_{\mu\nu\lambda}=s_1a_\lambda\hat{M}_{\mu\nu}+t_1(a_\mu\hat{M}_{\nu\lambda}-a_\nu\hat{M}_{\mu\lambda})+u_1a^\rho(\eta_{\nu\lambda}\hat{M}_{\rho\mu}-\eta_{\mu\lambda}\hat{M}_{\rho\nu}) \tag{4.17}$$

where s_1,t_1,u_1 are dimensionless parameters. In order to maintain the dimension of $\psi_{\mu\nu\lambda}$ as that of length, the inclusion of any higher order terms of $\hat{M}$ or $\hat{P}$ in the aforementioned ansatz would also raise the order of the deformation parameter a_μ. Our goal of maintaining terms linear in a_μ would have been hampered by this. The ansatz for $\psi_{\mu\nu\lambda}$ in this situation is thus expressed in its most generic form in (4.17). The following values of the parameters may be determined by performing a simple but laborious computation after inserting the ansatz back in (4.11,4.15):

$$t_1=-1; \qquad s_1=u_1=0$$

giving us the final form of $\psi_{\mu\nu\lambda}(\hat{P},\hat{M})$ as,

$$\psi_{\mu\nu\lambda} = -(a_\mu \hat{M}_{\nu\lambda} - a_\nu \hat{M}_{\mu\lambda}) \tag{4.18}$$

The ansatz for $\phi(\hat{P})$ and $\chi_{\mu\nu}(\hat{P},\hat{M})$ is next taken into account. It can be noted from (4.6) that $\chi_{\mu\nu}$ is dimensionless, hermitian, and doesn't have any special symmetry in its indices and must vanish in the commutative limit $a_\mu \to 0$. Additionally, because $\phi(\hat{P})$ is a *formal* scalar and has no dimensions, it can only be a function of $a_\mu \hat{P}^\mu$ and/or $a^2 \hat{P}_\mu \hat{P}^\mu$. It turns out that, we are compelled to examine higher orders in the deformation parameters in this case in order to obtain a consistent result due to the latter functional dependency. With these considerations, we make the following ansatz:

$$\phi(\hat{P}) = s_2 a^\mu \hat{P}_\mu + (1 + t_2 a^2 \hat{P}^2)^n; \quad \chi_{\mu\nu} = s_3 a_\mu \hat{P}_\nu + u_3 a_\nu \hat{P}_\mu + t_3 a^\lambda (\hat{P}_\lambda \hat{M}_{\mu\nu} + \hat{M}_{\mu\nu} \hat{P}_\lambda) \tag{4.19}$$

Note that, $\phi(P) \to 1$ and $\chi_{\mu\nu} \to 0$ for $a^\mu \to 0$, which is required in order to produce proper commutative limit. Here $s_2, t_2, n, s_3, u_3, t_3$ are dimensionless parameters. Now putting $\chi_{\mu\nu}$ back in (4.14), we get $t_3 = 0$, giving us $\chi_{\mu\nu} = s_3 a_\mu \hat{P}_\nu + u_3 a_\nu \hat{P}_\mu$.
Substituting the ansatz of $\phi(\hat{P})$ and $\chi_{\mu\nu}$ in (4.16), we get

$$s_2 = 1, \;\; s_3 = 1, \;\; u_3 = 0.$$

Thus $\chi_{\mu\nu}$ is now exactly solved and the ansatz for $\phi(\hat{P})$ reduces to:

$$\chi_{\mu\nu} = a_\mu \hat{P}_\nu; \qquad \phi(\hat{P}) = a.\hat{P} + (1 + t_2 a^2 \hat{P}^2)^n \tag{4.20}$$

Finally, substituting $\phi(\hat{P})$ and $\chi_{\mu\nu}$ (4.20), in (4.10), we get

$$n = \frac{1}{2} \text{ and } t_2 = 1.$$

Therefore, we have finally identified the deformed brackets that would enable us to derive the deformed transformation using Poincare generators, as

$$[\hat{M}_{\mu\nu}, \hat{X}_\rho] = i(\eta_{\nu\rho}\hat{X}_\mu - \eta_{\mu\rho}\hat{X}_\nu) - i(a_\mu \hat{M}_{\nu\rho} - a_\nu \hat{M}_{\mu\rho}) \tag{4.21}$$

$$[\hat{P}_\mu, \hat{X}_\nu] = -i\eta_{\mu\nu}\phi(\hat{P}) + i a_\mu \hat{P}_\nu; \quad \phi(\hat{P}) = a^\mu \hat{P}_\mu + \sqrt{1 + a^2 \hat{P}^2} \tag{4.22}$$

It is possible to verify that the aforementioned deformed commutators do in fact reproduce the standard brackets in the commutative limit $a_\mu \to 0$ limit, i.e.

$$[\hat{M}_{\mu\nu}, \hat{X}_\rho] = i(\eta_{\nu\rho}\hat{X}_\mu - \eta_{\mu\rho}\hat{X}_\nu); \qquad [\hat{P}_\mu, \hat{X}_\nu] = -i\eta_{\mu\nu} \tag{4.23}$$

So the deformed transformations compatible with the space-time algebra (4.1) can now be written as

$$\text{Under deformed translation: } \delta\hat{X}^\mu = i\epsilon^\alpha [\hat{P}_\alpha, \hat{X}^\mu] = \epsilon^\mu \phi(\hat{P}) - (\epsilon_\nu a^\nu)\hat{P}^\mu \tag{4.24}$$

$$\text{Under deformed Lorentz transformation: } \delta\hat{X}^\mu = \frac{i}{2}\omega^{\alpha\beta}[\hat{M}_{\alpha\beta}, \hat{X}^\mu] = \omega^\mu{}_\alpha \hat{X}^\alpha + \omega^{\alpha\beta} a_\alpha \hat{M}_\beta{}^\mu \tag{4.25}$$

Again in the commutative limit, they reduce to the typical transformations listed in (4.2).
The coordinate operators X^μ's reveal non-vector like transformations under the deformed Poincare transformations, as can be seen from (4.24, 4.25). It is simple to verify once more that these modi-

fications preserve the structure of coordinate algebra (4.1). In other words, under (4.24, 4.25), the structures of (4.1) remain stable.
It should be observed that while the Poincare algebra itself is not distorted, the deformed structure of the space-time commutator (4.1) causes the action of the Poincare generators on the module i.e. the algebra produced by the coordinates $\hat{X}^\mu$, to be severely deformed. That is to say, the undeformed character of the $\mathfrak{iso}(1,3)$ Lie algebra does not reflect the type of deformation. However, as we will show in the next section, that this deformed action has its roots in the deformed representation of the Lorentz generators $\hat{M}_{\mu\nu}$'s, when represented in terms of $\hat{X}_\mu$ and $\hat{P}_\mu$.

4.1.1 Deformed co-algebra and the construction of Heisenberg double

As we have seen in the last sub-section, the symmetry underlying the κ-Minkowski space-time $\hat{\mathcal{M}}$ is captured through deformed actions (4.24,4.25) of Lorentz generators. Remarkably, despite this deformation, the algebraic sector of the deformed symmetry remains unchanged compared to the undeformed Poincaré algebra. Naturally, a pertinent question arises: how does the deformed action affect the dynamics of single and/or multiparticle systems? As evidenced by a substantial body of literature [122,125,128,130], wherein the authors have diligently calculated the deformed co-algebraic structures, specifically the deformed coproducts (Δ), deformed antipodes (S), and co-units (ϵ). Although these co-algebraic structures may not bear direct relevance to our study of the kinematics and dynamics of a single-particle system, they can be employed to provide an alternative derivation of the deformed Heisenberg algebra (4.22) through the construction of a Heisenberg double within the framework of a Hopf algebroid. In this endeavor, we shall primarily follow the methodologies outlined in [57]. To embark on our exploration, let us present the expressions of the coalgebraic structures as documented in the literature.

$$\Delta(\hat{P}_\mu) = \hat{P}_\mu \otimes \phi + \mathbf{1} \otimes \hat{P}_\mu - a_\mu(\hat{P}_\nu\phi^{-1}) \otimes P^\nu + \frac{a_\mu}{2}(F(P)\phi^{-1}) \otimes (a.\hat{P}) \tag{4.26}$$

$$\Delta(\hat{M}_{\mu\nu}) = \hat{M}_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes \hat{M}_{\mu\nu} + a_\mu\Big(\hat{P}^\lambda - \frac{a^\lambda}{2}F(\hat{P})\Big)\phi^{-1} \otimes \hat{M}_{\lambda\nu} - a_\nu\Big(\hat{P}^\lambda - \frac{a^\lambda}{2}F(\hat{P})\Big)\phi^{-1} \otimes \hat{M}_{\lambda\mu} \tag{4.27}$$

$$F(\hat{P}) = \frac{2}{a^2}\Big(1 - \sqrt{1 + a^2\hat{P}^2}\Big)$$

We have provided a brief derivations of these coproducts in the Appendix-D [3]. And the deformed antipodes $S(\hat{M}_{\mu\nu})$ and $S(\hat{P}_\mu)$ are given by

$$S(\hat{P}_\mu) = \Big(- \hat{P}_\mu + ia_\mu(\hat{P}_\alpha - \frac{ia_\alpha}{2}F(\hat{P}))\hat{P}^\alpha\Big)\phi^{-1} \tag{4.28}$$

$$S(\hat{M}_{\mu\nu}) = -\hat{M}_{\mu\nu} + ia_\mu\Big(\hat{P}_\alpha - \frac{ia_\alpha}{2}F(\hat{P})\Big)\hat{M}_{\alpha\nu} - ia_\nu\Big(\hat{P}_\alpha - \frac{ia_\alpha}{2}F(\hat{P})\Big)\hat{M}_{\alpha\mu} \tag{4.29}$$

fulfilling,

$$m\Big[(\mathbf{1} \otimes S)\Delta\Big] = m\Big[(S \otimes \mathbf{1})\Delta\Big] = \eta \circ \epsilon$$

[3] Actually, it will become clear in the sequel that this exercise is tantamount to a verification of the self-consistency of the deformed Heisenberg algebra (4.22). This is because, (4.22) is used in Appendix-D to derive the coproduct $\Delta(\hat{P}_\mu)$, $\Delta(M_{\mu\nu})$ (4.21),(4.22). On the other hand, these coproducts are used in this sub-section to derive (4.22).

where η is unit of the algebra and ϵ is the co-unit given by

$$\epsilon(\hat{P}_\mu) = \epsilon(\hat{M}_{\mu\nu}) = 0 \tag{4.30}$$

which remains undeformed. One can observe at this stage that appropriate commutative limits i.e. the so called primitive forms of these co-algebraic structures are easily obtained in limit $a_\mu \to 0$.

$$\Delta(\hat{P}_\mu) \to \Delta_0(\hat{P}_\mu) = \hat{P}_\mu \otimes \mathbf{1} + \mathbf{1} \otimes \hat{P}_\mu \tag{4.31}$$

$$\Delta(\hat{M}_{\mu\nu}) \to \Delta_0(\hat{M}_{\mu\nu}) = \hat{M}_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes \hat{M}_{\mu\nu} \tag{4.32}$$

$$S(\hat{P}_\mu) \to S_0(\hat{P}_\mu) = -\hat{P}\mu \tag{4.33}$$

$$S(\hat{M}_{\mu\nu}) \to S_0(\hat{M}_{\mu\nu}) = -\hat{M}_{\mu\nu} \tag{4.34}$$

First observe that the abelian sub-algebra $[\hat{P}_\mu, \hat{P}_\nu] = 0$ (4.3) indicates that the space $\mathcal{J}$ of abelian space-time translation generators $\hat{P}_\mu \in \mathcal{J} \subset \mathfrak{iso}(1,3)$ in the commutative ($a_\mu \to 0$) limit ($\hat{\mathcal{M}} \to \mathcal{M}$) with $\hat{X}_\mu \in \hat{\mathcal{M}} \to \hat{q}_\mu \in \mathcal{M}$ can be used to describe the standard quantum mechanical phase space, associated with undeformed Heisenberg algebra ($[\hat{q}_\mu, \hat{q}_\nu] = 0 = [\hat{P}_\mu, \hat{P}_\nu]; [\hat{P}_\mu, \hat{q}_\nu] = -i\eta_{\mu\nu}$), in terms of a smash product $\mathcal{H}_0 := \mathcal{U}(\mathcal{J}) \# \mathcal{U}(\mathcal{M})$, defining Heisenberg double with undeformed Heisenberg Hopf algebroid structure [4]. Note that here the generators of the 4-momenta which are dual to q's can act on $\mathcal{M}$. And in this Hopf algebroid, we have two abelian Hopf algebras, given as functions of $\hat{q}^\mu$ and $\hat{P}_\mu$, which are dual to each other. The coalgebra sector of $\hat{q}_\mu$'s are clearly primitive i.e. undeformed. In particular, the coproduct of $\hat{q}_\mu$ is

$$\Delta_0(\hat{q}_\mu) = \hat{q}_\mu^{(1)} \otimes \hat{q}_\mu^{(2)} = \hat{q}_\mu \otimes \mathbf{1} + \mathbf{1} \otimes \hat{q}_\mu \tag{4.35}$$

where we have made use of the Sweedler's notation. The primitive coproduct of $\hat{P}_\mu$ (4.31) can likewise be expressed as $\Delta_0(\hat{P}_\mu) = \hat{P}_\mu^{(1)} \otimes \hat{P}_\mu^{(2)}$. Using this notation, we can define the cross-multiplication rules for $\mathcal{H}_0$ as [130]

$$\hat{P}_\mu \hat{q}_\nu = \hat{q}_\nu^{(1)} \langle \hat{P}_\mu^{(1)}, \hat{q}_\nu^{(2)} \rangle \hat{P}_\mu^{(2)} \tag{4.36}$$

where $\langle ., . \rangle$ represents canonical duality pairing and is defined as,

$$\langle \hat{P}_\mu, \hat{q}_\nu \rangle := \hat{P}_\mu \triangleright \hat{q}_\nu = [\hat{P}_\mu, \hat{q}_\nu] \triangleright \mathbf{1} = -i\eta_{\mu\nu}. \tag{4.37}$$

And this, in a Hopf-algebraic scheme, corresponds to the binary duality map

$$\mathcal{J} \otimes \mathcal{M} \to \mathbb{C}$$

$$p \otimes q \to \langle p, q \rangle$$

Note that, here we are just dealing with translational abelian subalgebra $\mathcal{J}$ of Poincare algebra: $\mathcal{J} \subset \mathfrak{iso}(1,3)$. Equation (4.37) needs to be augmented with the following actions

$$\hat{q}_\mu \triangleright \mathbf{1} = \hat{q}_\mu, \quad \hat{P}_\mu \triangleright \mathbf{1} = 0 \tag{4.38}$$

Now to capture NC phase space, we need to construct the deformed smash product $\mathcal{H} := \mathcal{U}(\hat{\mathcal{J}}) \# \mathcal{U}(\hat{\mathcal{M}})$, where we need to essentially replace the primitive coproduct of abelian generators (4.3) $\hat{P}_\mu$ by the de-

[4] As we have shown later, that $\hat{q}_\mu$ in our case can be related to $\hat{X}_\nu$ through a momentum dependent non-singular matrix (4.82), while the momentum undergoes no deformation.

formed one: $\Delta_0(\hat{P}_\mu) \to \Delta(\hat{P}_\mu)$ (4.21). On the other hand, the actions of $\hat{P}_\mu$ and $\hat{X}_\mu$, along with the duality retain their same forms given by

$$\hat{X}_\mu \triangleright \mathbf{1} = \hat{X}_\mu, \quad \langle \hat{P}_\mu, \hat{X}_\nu \rangle := \hat{P}_\mu \triangleright \hat{X}_\nu = [\hat{P}_\mu, \hat{X}_\nu] \triangleright \mathbf{1} = -i\eta_{\mu\nu} \tag{4.39}$$

Not only that, the cross multiplication in $\mathcal{H}$ also retains the same form (4.36)

$$\hat{P}_\mu \hat{X}_\nu = \hat{X}_\nu^{(1)} \langle \hat{P}_\mu^{(1)}, \hat{X}_\nu^{(2)} \rangle \hat{P}_\mu^{(2)} \tag{4.40}$$

except that we now need to use the primitive co-product for $\hat{X}_\mu$ i.e. $\Delta_0(\hat{X}_\mu) = \hat{X}_\mu^{(1)} \otimes \hat{X}_\mu^{(2)} = \hat{X}_\mu \otimes \mathbf{1} + \mathbf{1} \otimes \hat{X}_\mu$ (D.10) here, rather than the deformed coproduct $\Delta(\hat{X}_\mu)$ (D.9). To understand the reason behind this, consider the following pairing with the rule $\langle a \otimes b, c \otimes d \rangle = \langle a, c \rangle \langle b, d \rangle$

$$\begin{aligned} \langle \Delta \hat{P}_\mu, \hat{X}_\nu \otimes \hat{X}_\rho \rangle &= \langle \hat{P}_\mu, \hat{X}_\nu \rangle \langle \phi, \hat{X}_\rho \rangle - a_\mu \Big\langle \phi^{-1} \Big(\hat{P}_\lambda - \frac{a_\lambda}{2} F \Big), \hat{X}_\nu \Big\rangle \langle \hat{P}^\lambda, \hat{X}_\rho \rangle \\ &= -i\eta_{\mu\nu} a_\rho + i a_\mu \eta_{\rho\nu} \end{aligned} \tag{4.41}$$

where we have used $\langle f(\hat{P}), \hat{X}_\rho \rangle = f(\hat{P}) \triangleright \hat{X}_\rho = [f(\hat{P}), \hat{X}_\rho] \triangleright \mathbf{1}$ for an arbitrary function $f(\hat{P})$ and $\phi(\hat{P}) \triangleright \mathbf{1} = \mathbf{1}$ for $\phi(\hat{P})$ given in (4.20).
Now anti-symmetrizing the relation (4.41) with respect to the indices ν, ρ, we can write, using (4.1)

$$\langle \Delta(\hat{P}_\mu), \hat{X}_\nu \otimes \hat{X}_\rho - \hat{X}_\rho \otimes \hat{X}_\nu \rangle = \langle \hat{P}_\mu, [\hat{X}_\nu, \hat{X}_\rho] \rangle = i(\eta_{\mu\rho} a_\nu - \eta_{\mu\nu} a_\rho) \tag{4.42}$$

This shows that the commutator algebra (4.1) is dual to the coproduct $\Delta(\hat{P}_\mu)$. One can easily check, at this stage, that this duality will not hold if we were to make use of the undeformed i.e. primitive coproduct $\Delta_0(\hat{P}_\mu)$ (4.31) i.e. $\langle \Delta_0(\hat{P}_\mu), \hat{X}_\nu \otimes \hat{X}_\rho - \hat{X}_\rho \otimes \hat{X}_\nu \rangle \neq \langle \hat{P}_\mu, [\hat{X}_\nu, \hat{X}_\rho] \rangle$. On the other hand, the vanishing nature of the pairing

$$\langle \hat{P}_\mu \otimes \hat{P}_\nu - \hat{P}_\nu \otimes \hat{P}_\mu, \Delta_0(\hat{X}_\lambda) \rangle = \langle [\hat{P}_\mu, \hat{P}_\nu], \hat{X}_\lambda \rangle = 0 \tag{4.43}$$

indicates that the vanishing commutator of $\hat{P}_\mu$'s (4.3) is dual to the primitive coproduct $\Delta_0(\hat{X}_\mu)$. Again this duality will not hold if we were to use the deformed coproduct $\Delta(\hat{X}_\mu)$ (D.9) as this involves $\hat{P}_\mu$'s in its expansion [5].
These important observations deserve to be emphasized once more. So to put it in other words, the commutator algebra (4.1) involving space-time coordinates $\hat{X}_\mu$ is dual (4.41) to *deformed* coproduct of momenta $\Delta(\hat{P}_\mu)$ (4.40). In contrast, the vanishing commutator algebra (4.3) involving momenta $\hat{P}_\mu$ is dual (4.43) to the *undeformed* i.e. primitive coproduct of $\hat{X}_\mu$ i.e. $\Delta_0(\hat{X}_\mu)$ (D.10).
So finally, making use of (4.40) we can write, following [57],

$$\begin{aligned} [\hat{P}_\mu, \hat{X}_\nu] &= \hat{X}_\nu^{(1)} \langle \hat{P}_\mu^{(1)}, \hat{X}_\nu^{(2)} \rangle \hat{P}_\mu^{(2)} - \hat{X}_\nu \hat{P}_\mu \\ &= -i\eta_{\mu\nu} + m\Big[(\Delta - \Delta_0)(\hat{P}_\mu)(\triangleright \otimes \mathbf{1})(\hat{X}_\nu \otimes \mathbf{1}) \Big] \end{aligned}$$

A straightforward computation the readily yields

$$[\hat{P}_\mu, \hat{X}_\nu] = -i\eta_{\mu\nu}\phi + i a_\mu \hat{P}_\nu \tag{4.44}$$

[5]This coproduct $\Delta(\hat{X}_\mu)$ (D.9) will correspond to the Hopf algebroid $\tilde{\mathcal{H}} := \mathcal{U}(\mathfrak{iso})(1,3)\#\mathcal{U}(\hat{\mathcal{M}})$, where the algebra $\mathcal{A}$ is associated with enlarged basis $(\hat{X}_\mu, \hat{P}_\mu, \hat{M}_{\mu\nu})$ (Appendix-D) [57]. Here, of course, we are not concerned with that.

reproducing the deformed Heisenberg algebra (4.22).

Now with this, the co-algebra sector gets deformed, deforming the Hopf algebra structure as a whole. Thus, it is clear that the two particle sector of the symmetry generators' action is significantly distorted. Naturally, the following question arises: what deformation, if any, exists in the one particle sector itself and how to recognise it? By constructing a dynamical model, namely the Lagrangian of a relativistic spin-less free particle in κ deformed space-time, which possesses the same symmetries as those of the space-time itself, we attempt to provide a solution to this question in the next section. It will also be demonstrated that the model generates non-trivial momentum space geometry, which in turn produces a dispersion relation that encodes the deformation in the one particle sector.

4.2 Construction of a dynamical model invariant under deformed symmetries

We've already seen that deformation has no discernible impact in the one particle sector of κ Minkowski space because the undeformed $\mathfrak{iso}(1,3)$ algebra yields an undeformed Casimir $P^2 = P_\mu P^\mu$, which is anticipated to be adequate to label spin-less one particle states by assigning a mass $m\,(P^2 = m^2)$ to it. However, a modification in the motion of a relativistic free particle, on the other hand, may result from a deformed mass-shell condition caused by a curved momentum space. In this part, we demonstrate how to build a dynamical model that respects deformed symmetries and that produces both the classical κ Minkowski algebra and the phase space algebra resulting from its symplectic structure, as well as the deformed mass-shell condition that affects dynamics. As we shall see that the mass-shell condition has no bearing on the symplectic structure, so we first assume it to be a generic deformation of the typical mass-shell condition. Later, we derive its precise form from the geodesic distance in a curved momentum space, which appears as a bi-product because space-time isn't commutative.

Here, using a purely algebraic method, we provide a systematic procedure to construct a first-order version of the Lagrangian L_f^τ to represent a relativistic free particle. We require that the Lagrangian must yield a symplectic structure, which is same as the classical version of the phase space algebra of κ Minkowski space, and that it must be unaffected under the deformed symmetry transformations (4.24,4.25) of κ Minkowski space-time. On the way, it will be clear how the algebraic and dynamical methods are consistent with one another. It is gratifying to see that the Poincare generators accountable for the relevant deformed transformations are really produced by the symmetry generators determined using Noether's method, i.e. by varying the above Lagrangian.

The Lie algebraic type of noncommutativity between the coordinates (4.1) and the intricate structure of the deformation in the sector of the phase-space algebra (4.22) makes it challenging to write the Lagrangian of a relativistic free particle that also respects the deformed symmetries of a κ Minkowski space-time. This was previously done for Snyder space-time [131, 132] differently. Here, we propose a novel yet simple way to build the required Lagrangian, that we will discuss step by step in the following paragraphs.

As a constrained system, a first-order form of the Lagrangian of a relativistic free particle in κ Minkowski space must yield Dirac brackets compatible with the coordinate and phase space algebras associated to the commutators (4.1, 4.21, 4.22), and may be considered as the classical counterpart of the respective commutators. As a result, we begin by demoting the operators $\hat{X}, \hat{P}$ to the classical commuting variables X, P, which satisfy the necessary Dirac brackets derived from the commutators (4.1) and (4.22) by proper substitution. Therefore, we essentially operate in the limit $\hbar \to 0$. It follows that neither quantum effects nor effects of non-commutativity in the form of a fundamental length scale (such as $l_P = \sqrt{\hbar G}\,(c=1)$) are anticipated to persist in this limit. Specifically, the non-commutative parameter a^μ that was assumed to be of the order of the Planck length scale l_P vanishes in this limit. However, it should be noted that constructing the Dirac bracket from the commutator bracket necessitates the following identification:

$$[\hat{f}, \hat{g}] \longrightarrow \{f, g\}_{D.B} = \lim_{\hbar \to 0} \; \frac{1}{i\hbar}\,[f, g] \tag{4.45}$$

where $\hbar$ is reinstated in the commutator brackets. Thus we have to effectively replace $a^\mu \to \frac{a^\mu}{\hbar} =: \mathfrak{a}^\mu$ in all the relations (4.1) and (4.22) and write,

$$\{X_\mu, X_\nu\}_{D.B} = \mathfrak{a}_\mu X_\nu - \mathfrak{a}_\nu X_\mu = \theta_{\mu\nu}; \quad \{P_\mu, X_\nu\}_{D.B} = -\eta_{\mu\nu}\left[\mathfrak{a}.P + \sqrt{1+\mathfrak{a}^2P^2}\right] + \mathfrak{a}_\mu P_\nu; \quad \{P_\mu, P_\nu\}_{D.B} = 0 \tag{4.46}$$

Now $\mathfrak{a}^\mu$ will be of the order of $\sqrt{\frac{G}{\hbar}}$. Even though the Planck length $\sqrt{G\hbar}$ fails to survive the limit $\hbar \to 0$, it is possible to hold the ratio $\frac{G}{\hbar}$ fixed if both the limits $G \to 0$ and $\hbar \to 0$ are taken at the same time. This gives us a mass scale of $\sqrt{\frac{G}{\hbar}} \sim \frac{1}{\kappa}$, which could potentially be measured through experiment but is assumed to be the inverse of the Planck mass m_P. This may be viewed as a regime of quantum gravity, where the only inherent scale is the mass scale m_P. In light of this, one does not expect to observe any effects of the NC character of space-time resulting from a length scale, but one can expect to observe the effects of finite mass scale via the appearance of curved momentum space [52], which is then anticipated to change the dispersion relation even for a single relativistic particle.

In the literature [97,98], there is a well-established algorithm for getting Dirac brackets from a given constrained form of Lagrangian. But in this case, since we have the anticipated Dirac brackets (4.46), we follow the route backwards to obtain the first-order Lagrangian in the following manner:

- Recall that the Poission bracket $\{.,.\}$ may be used to express the Dirac bracket (D.B) $\{.,.\}_{DB}$ among the phase space variables as

$$\{\xi^{(a)}_\mu, \xi^{(b)}_\nu\}_{D.B} = \{\xi^{(a)}_\mu, \xi^{(b)}_\nu\} - \{\xi^{(a)}_\mu, \Sigma^{(c)}_\alpha\}(\Lambda^{-1})^{\alpha\beta}{}_{cd}\{\Sigma^{(d)}_\beta, \xi^{(b)}_\nu\}; \quad a, b = 1, 2; \mu, \nu = 0, 1, 2, 3 \tag{4.47}$$

  Both X_μ and P_μ are thought of being configuration space variables in a first-order Lagrangian of an extended configuration space. It is, therefore, appropriate to refer to them as $\xi^{(1)}_\mu = X_\mu$ and $\xi^{(2)}_\mu = P_\mu$. Two sets of constraints associating canonical momenta with functions of the enlarged configuration space variables are produced by a specific first-order Lagrangian, taken to be of the following form

$$L = -g_\mu(X, P)\dot{P}^\mu - f_\mu(X, P)\dot{X}^\mu - H \tag{4.48}$$

  dependent on X, P, and they have the following general structural form:

$$\Sigma^1_\mu = \Pi^X_\mu + f_\mu(X, P) \approx 0; \qquad \Sigma^2_\mu = \Pi^P_\mu + g_\mu(X, P) \approx 0. \tag{4.49}$$

  where $\Pi^X_\mu = \frac{\partial L}{\partial \dot{X}^\mu}$ and $\Pi^P_\mu = \frac{\partial L}{\partial \dot{P}^\mu}$ as the canonical momenta conjugate to X_μ and P_μ, fulfilling the following relations,

$$\{X_\mu, \Pi^X_\nu\} = \eta_{\mu\nu} = \{P_\mu, \Pi^P_\nu\} \tag{4.50}$$

We must specifically determine $f_\mu(X,P)$ and $g_\mu(X,P)$, which are appropriate functions of configuration space, in order to build the Lagrangian of a free relativistic particle in κ-Minkowski space. Additionally, Λ^{-1} in (4.47) is the inverse of the constraint matrix Λ, which may be obtained by using (4.49) as

$$(\Lambda_{\mu\nu})^{ab} = \{\Sigma^{(a)}_\mu, \Sigma^{(b)}_\nu\} \tag{4.51}$$

- Since we already know the Dirac bracket relationships between the phase-space variables provided by (4.46), we can quickly get the inverse of the constraint matrix Λ from (4.47) by varying the index a of $\xi^{(a)}_\mu$, as shown below:

$$(\Lambda^{-1})^{\mu\nu}{}_{ab} = \begin{pmatrix} \theta^{\mu\nu} & \eta^{\mu\nu}\phi(P) - \mathfrak{a}^\nu P^\mu \\ -\eta^{\mu\nu}\phi(P) + \mathfrak{a}^\mu P^\nu & 0 \end{pmatrix} \tag{4.52}$$

- The corresponding inverse matrix Λ can be obtained as,

$$\Lambda_{\mu\nu,ab} = \phi^{-1}(P) \begin{pmatrix} 0 & -\eta_{\mu\nu} - t(P)\mathfrak{a}_\mu P_\nu \\ \eta_{\mu\nu}(P) + t(P)\mathfrak{a}_\nu P_\mu & \phi^{-1}(P)\Big[\theta_{\mu\nu} + t(P)(\theta_{\mu\alpha}\mathfrak{a}^\alpha P_\nu - \theta_{\nu\alpha}\mathfrak{a}^\alpha P_\mu)\Big] \end{pmatrix} \tag{4.53}$$

where $t(P) = \frac{1}{\phi(P) - \mathfrak{a}.P}$. One can indeed verify the identity $\Lambda_{\mu\nu,ab}(\Lambda^{-1})^{\nu\lambda}{}_{bc} = \delta_\mu{}^\lambda\ \delta_{ac}$ holds. The precise forms of both functions $f_\mu(X,P)$ and $g_\mu(X,P)$ in the constraints (4.49) must be found, and the constraints must satisfy the relation (4.51). We get at the following set of solutions for the phase space function $f_\mu(X,P)$ and $g_\mu(X,P)$ in (4.48) and (4.49)

$$f_\mu(P) = 0; \qquad g_\mu(X,P) = \phi^{-1}(P)\Big[X_\mu + \frac{(\mathfrak{a}.X)P_\mu}{\phi(P) - \mathfrak{a}.P}\Big] \tag{4.54}$$

using simple observations along with some informed guesses. By inserting the solutions (4.54) back into (4.48) and (4.49), one can verify that the relations in (4.51) are in fact fulfilled and that the solution is unique up to a total time derivative (see Appendix-D)

- We may retrieve the explicit forms of the canonical momenta, after the constraints have been identified, as follows,

$$\Pi^X_\mu = -f_\mu = 0; \qquad \Pi^P_\mu = -g_\mu = -\phi^{-1}(P)\Big[X_\mu + \frac{(\mathfrak{a}.X)P_\mu}{\phi(P) - \mathfrak{a}.P}\Big] \tag{4.55}$$

The desired Lagrangian (4.48) of a relativistic free particle in first order form may be expressed as follows,

$$L^\tau_f = -\phi^{-1}(P)\Big[X_\mu + \frac{(\mathfrak{a}.X)P_\mu}{\phi(P) - \mathfrak{a}.P}\Big]\dot{P}^\mu - e(f(P^2) - f(m^2)) \tag{4.56}$$

Here, the expected deformed mass-shell condition $f(P^2) - f(m^2) = 0$ is enforced by the Lagrangian multiplier e and τ is the system's evolution parameter. While "m" can be understood as the "bare" mass occurring in the eigenvalue equation of the Casimir operator $P^2 = m^2$, $M = \sqrt{f(m^2)}$ is to be identified in the sequel, with the measured renormalized mass of the particle. We assume that $m > 0$ and that P^μ is a time-like vector. Using this version of the Lagrangian, one can now do the Hamiltonian analysis and verify that the desired classical realisation of the κ Minkowski phase-space algebra (4.46) is indeed reproduced (see Appendix-D).

Note that, due to our expectation that the d'Alembertian will commute with all the generators of the $\mathfrak{iso}(1,3)$ algebra, we have opted to use a deformed mass-shell condition in which the associated d'Alembertian operator is selected to be a function of P^2 as $f(P^2)$ instead of just P^2 (4.3). The Hamiltonian H may be clearly recognized as,

$$H = e(f(P^2) - M^2) \tag{4.57}$$

The symplectic structure (4.46) turns out to be unaffected by the precise form of the function f, but the temporal evolution of the system, which is now understood to represent the unfolding of gauge transformations produced by the first class constraint $f(P^2) - M^2 \approx 0$ does depend on f. By relating the d'Alembertian to the squared geodesic distance between a preferred origin, taken that corresponds to the vacuum $P^\mu = 0$, and a point p with coordinate P^μ in momentum space $\mathcal{P}$, it will be possible to precisely identify the functional form of f in the following section. We will discover that it is deformed as a result of the curved and non-trivial geometry of momentum space.

4.2.1 Deformed mass-shell condition

In the special theory of relativity, with flat momentum space $\mathcal{P}_0$ the dispersion relation can be understood as the square of the distance along a straight line between a particular point p having coordinates P^μ in $\mathcal{P}_0$ and an origin that is chosen to correspond to the ground state, coordinatized as $P^\mu = 0$ (Fig. 4.1).

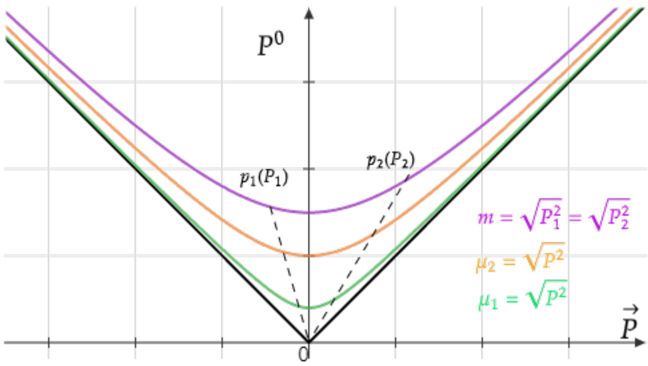

Figure 4.1: Mass-shell diagram

The diagram depicts one parameter family of hyperboloids, parameterized by eigen-values of the Casimir $\sqrt{P^2} = \mu$, covering the forward light cone in the flat momentum space $\mathcal{P}_0$. A sample of 3 hyperboloids has been shown. Here the dashed lines indicate straight line paths connecting the origin to the points on the final hyperboloid which is the mass-shell $\sqrt{P^2} = m$ with momentum coordinates P_1^μ, P_2^μ in $\mathcal{P}_0$.

The square of distance in Minkowski space, which is $C = \eta_{\alpha\beta}P^\alpha P^\beta = P^2 = m^2$, is generalised

[60,133,134] for curved momentum space to

$$C = [D(0,P)]^2 = \Big(\int_0^P \sqrt{g_{\mu\nu}(p) dp^\mu dp^\nu} \Big)^2 = \Big(\int_0^\tau d\tau' \sqrt{g_{\mu\nu}(p) \dot{p}^\mu \dot{p}^\nu} \Big)^2 ; \qquad \dot{p}^\mu = \frac{dp^\mu}{d\tau'} \tag{4.58}$$

where the momentum space metric is $g_{\mu\nu}(p)$. To generate a coordinate chart for the points in the forward light cone of the curved momentum space $\mathcal{P}$, we once again use the P_μ's of the flat momentum space $\mathcal{P}_0$.[6] We are going to begin with the task of identifying $g_{\mu\nu}(P)$. First, notice that we may recast the Lagrangian one-form in (4.56) in the following manner for more concise expression:

$$L_f^\tau d\tau = -X^\alpha E(P)_\alpha{}^\mu dP_\mu - e(f(P^2) - M^2) d\tau; \qquad E(P)_\alpha{}^\mu = \phi^{-1} \Big(\delta_\alpha{}^\mu + \frac{\mathfrak{a}_\alpha P^\mu}{\phi - \mathfrak{a}.P} \Big) \tag{4.59}$$

Additionally, the occurrence of this $E(P)_\alpha{}^\mu$ may be linked to (4.22), where its inverse occurs as

$$\{X^\alpha, P_\mu\} = (E^{-1}(P))^\alpha{}_\mu; \quad (E^{-1}(P))^\alpha{}_\mu = \delta^\alpha{}_\mu \phi(P) - \mathfrak{a}_\mu P^\alpha; \quad (E^{-1}(P))^\alpha{}_\mu E(P)^\mu{}_\beta = \delta^\alpha{}_\beta \tag{4.60}$$

in the corresponding Dirac bracket. In comparison, the Lagrangian one form of a relativistic free particle in flat momentum space is provided by

$$\mathcal{L}_f^\tau d\tau = -x^\alpha \delta_\alpha{}^\beta dp_\beta - H d\tau \tag{4.61}$$

It is obvious that the curved nature of momentum space ($\mathcal{P}$) geometry is expected to arise from the non-trivial components of $E(P)$ (4.59) or $E^{-1}(P)$ (4.60), which are now interpreted as the components of of vielbeins in the momentum space. It logically implies that there will be some modification from the flat momentum space ($\mathcal{P}_0$) line element

$$dS^2 = \eta_{ab} dP^a dP^b \tag{4.62}$$

to some modified expression in curved momentum space entailing suitable insertions of E or E^{-1} matrices, where the Latin indices are to indicate the flat Lorentz indices. Before continuing, it should be noted that the cotangent space $T_p^*(\mathcal{P}_0)$ at an arbitrarily chosen point $p \in \mathcal{P}_0$ is inherently holonomic in nature and dP^a's in the aforementioned statement serve as its orthonormal basis. It is only possible to generate equivalent orthonormal basis e^a's for the cotangent space $T_p^*(\mathcal{P})$ for $p \in \mathcal{P}$ by contracting the indices of the E^{-1} matrix as

$$e^a = (E^{-1})^a{}_\mu dP^\mu \tag{4.63}$$

which, generally speaking, is non-holonomic (i.e. non-exact form) in nature $de^a \neq 0$. To distinguish orthonormal and holonomic basis elements in this instance, we have once more employed Latin and Greek indices. Also note that $E_a{}^\mu \to \delta_a{}^\mu$ when either $\mathfrak{a}_\mu$ or $P_\mu \to 0$. By replacing dP^a in (4.62) with e^a in (4.21), it is now simple to construct the line element corresponding to the curved momentum space $\mathcal{P}$ as

$$dS^2 = \eta_{ab} (E^{-1})^a{}_\mu (E^{-1})^b{}_\nu dP^\mu dP^\nu \tag{4.64}$$

This helps us to just read-off the metric $\tilde{g}_{\mu\nu}(P)$ in $\mathcal{P}$ as

$$\tilde{g}_{\mu\nu}(P) = \eta_{ab} (E^{-1})^a{}_\mu (E^{-1})^b{}_\nu \tag{4.65}$$

[6] Remember that since we aren't working with massless particles here, we are excluding the null directions from our consideration.

so that

$$dS^2 = \tilde{g}_{\mu\nu}(P)dP^\mu dP^\nu \tag{4.66}$$

holds. Now substituting $(E^{-1})^a{}_\mu$ from (4.60), we get

$$\tilde{g}_{\mu\nu}(\tilde{P}) = \phi^2\eta_{\mu\nu} - \phi(\mathfrak{a}_\mu\tilde{P}_\nu + \mathfrak{a}_\nu\tilde{P}_\mu) + \mathfrak{a}_\mu\mathfrak{a}_\nu\tilde{P}^2 \tag{4.67}$$

the inverse of which is given as

$$\tilde{g}^{\mu\nu}(\tilde{P}) = \eta^{ab}E_a{}^\mu(\tilde{P})E_b{}^\nu(\tilde{P}) = \phi^{-2}\Big[\eta^{\mu\nu} + \frac{\mathfrak{a}^\mu\tilde{P}^\nu + \mathfrak{a}^\nu\tilde{P}^\mu}{\phi - \mathfrak{a}.\tilde{P}} + \frac{\mathfrak{a}^2\tilde{P}^\mu\tilde{P}^\nu}{(\phi-\mathfrak{a}.\tilde{P})^2}\Big]; \quad \tilde{g}^{\mu\nu}\tilde{g}_{\nu\rho} = \delta^\mu{}_\rho \tag{4.68}$$

The significance of using an overhead tilde over the metric $g_{\mu\nu}$ and P^μ's will become clear very soon. It is important to note that neither $\tilde{g}_{\mu\nu}$ nor $\tilde{g}^{\mu\nu}$ are covariant under Lorentz transformations, primarily due to the presence of $\mathfrak{a}^\mu$'s, which are not true vectors. As a result, they cannot even be considered proper as Lorentz tensors and the momentum space $\mathcal{P}$ cannot be identified as a genuine differentiable manifold equipped with a metric field $g_{\mu\nu}(P)$ that transforms covariantly under diffeomorphisms. Instead, $g_{\mu\nu}(P)$ represents a four-parameter family of deformed metrics, and $\mathcal{P}$ cannot be associated with a group manifold or any maximally symmetric space either, unlike the one in [52, 60], for instance, where only a single mass scale is involved. However, since the geodesic distance (4.31) in a curved manifold should remain invariant under diffeomorphisms, we cannot use the non-tensorial metric $\tilde{g}_{\mu\nu}(\tilde{P})$ (4.67) to compute the geodesic distance. In this scenario, to extract any meaningful information regarding the extremal distance, we can *formally* introduce a covariantly transforming metric $g_{\mu\nu}(P)$, where $\tilde{g}_{\mu\nu}(\tilde{P})$ (4.67) is treated as a specific form of the metric $g_{\mu\nu}(P)$ in a chosen inertial frame, referred to as the fiducial frame. In this formal treatment, we distinguish the corresponding quantities using overhead tildes. This formalism ensures the diffeomorphism invariance of dS^2 (4.66). However, to achieve this, we need to *formally* promote both the $\mathfrak{a}^\mu$'s (a quadruplet of scalars) and P^μ's (Lorentz four-vectors) to objects that transform as vectors under diffeomorphisms, as follows:

$$\mathfrak{a}^\beta \to \frac{\partial P^\mu}{\partial\tilde{P}^\beta}\mathfrak{a}^\beta; \qquad \tilde{P}^\beta \to \frac{\partial P^\mu}{\partial\tilde{P}^\beta}\tilde{P}^\beta. \tag{4.69}$$

Similarly, we do the same for their counterparts $\mathfrak{a}_\mu, P_\mu$ that are covariantly transforming such that the metric $g_{\mu\nu}(P)$ may be generated from $\tilde{g}_{\mu\nu}(\tilde{P})$ by requiring it to change covariantly as

$$\tilde{g}_{\mu\nu}(\tilde{P}) \to g_{\mu\nu}(P) = \frac{\partial\tilde{P}^\alpha}{\partial P^\mu}\frac{\partial\tilde{P}^\beta}{\partial P^\nu}\tilde{g}_{\alpha\beta}(\tilde{P}) \tag{4.70}$$

and the flat metric $\eta_{\alpha\beta}$ to transform likewise as $\eta_{\alpha\beta} \to G_{\mu\nu} = \frac{\partial\tilde{P}^\alpha}{\partial P^\mu}\frac{\partial\tilde{P}^\beta}{\partial P^\nu}\eta_{\alpha\beta}$. We shall restore the respective statuses of $\mathfrak{a}^\alpha, P^\alpha$ very soon.

For objects referred to in the fiducial frame, we use initial greek indices like α and β as subscripts or superscripts, and middle ones like μ, ν, etc. in any other frame. Further, at this *formal* level, we can see that

$$\tilde{P}^2 = \eta_{\alpha\beta}\tilde{P}^\alpha\tilde{P}^\beta = \mu^2 \to G_{\mu\nu}\frac{\partial P^\mu}{\partial\tilde{P}^\alpha}\frac{\partial P^\nu}{\partial\tilde{P}^\beta}\tilde{P}^\alpha\tilde{P}^\beta = \eta_{\alpha\beta}\tilde{P}^\alpha\tilde{P}^\beta = \mu^2 \tag{4.71}$$

and therefore μ^2 remains invariant and can be treated as a scalar under diffeomorphism as well. Consequently,

$$\tilde{g}_{\alpha\beta}(\tilde{P})\tilde{P}^\alpha\tilde{P}^\beta = \tilde{P}^2(1+\mathfrak{a}^2\tilde{P}^2) = \mu^2(1+\mathfrak{a}^2\mu^2) \tag{4.72}$$

is also diffeomorphism invariant. Here we have made use of the condition $\tilde{P}^2 = \eta_{\alpha\beta}\tilde{P}^\alpha\tilde{P}^\beta = \mu^2$, appropriate for the hyperboloid labelled by $\sqrt{\tilde{P}^2} = \mu$ (Fig. 4.1). This quantity (4.72) is specifically needed for our computation of geodesic distance which we shall discuss next. So it is evident that even if we were to compute this quantity (4.72) for any other choice of the fiducial frame, our results will not change.

When calculating the deformed dispersion relation, we must first determine the geodesic distance between the origin, which stands for the ground state and has the coordinate $P^\mu = 0$, and any other point p in the forward light cone in the energy-momentum space $\mathcal{P}$. While it is obvious that the geodesic must be time-like in the sense that the tangent vector at any point along the geodesic must be time-like, the intricate structure of $\tilde{g}_{\mu\nu}$ (4.67) makes it challenging to explicitly calculate the geodesic distance. Therefore, we will use a different method and employ the differential equation [135] fulfilled by the geodesic distance $D(P) := D(0, P)$ (4.58), which is given by

$$\partial^\mu D(P)\, g_{\mu\nu}(P)\, \partial^\nu D(P) = 1; \qquad \partial^\mu := \frac{\partial}{\partial P_\mu} \tag{4.73}$$

where we have assumed that $\mathcal{P}$ is a genuine Riemannian manifold[7] . Using $C = D^2$, where C is the d'Alembertian operator, one can express this equivalently as,

$$\partial^\mu C(P)\, g_{\mu\nu}(P)\, \partial^\nu C(P) = 4C \tag{4.74}$$

As previously stated, we now limit our search for C to the form $C = f(P^2)$. Geometrically, this simply implies that, if P_1 and P_2 are both members of the same hyperboloid, then $P_1^2 = P_2^2 = m^2$. To put it another way, we suppose that this equality holds true for both the flat space $\mathcal{P}_0$ and the curved space $\mathcal{P}$ (see Fig. 4.1). As a result, the distances between the origin ($P^\mu = 0$) and any other point in the same mass-shell remain equal. Additionally, this guarantees that the whole Lagrangian is Poincare invariant (4.56). Finally, by applying (4.72) and the substitution $C = f(P^2)$, we obtain,

$$M = \sqrt{C} = D(0, P) = \frac{1}{2}\int_0^{P^2=m^2} \frac{d(\mu^2)}{\sqrt{\mu^2(1+\mathfrak{a}^2\mu^2)}} = \int_0^m \frac{d\mu}{\sqrt{1+\mathfrak{a}^2\mu^2}}, \qquad \mathfrak{a}^2 = \eta_{\mu\nu}\mathfrak{a}^\mu\mathfrak{a}^\nu \tag{4.75}$$

At this stage, we can restore the respective original statuses of $\mathfrak{a}^\mu$ and P^μ. So far we have not imposed any condition on $\mathfrak{a}^2$, but now we will consider three different cases for $\mathfrak{a}_\mu$ being “null" (i.e. $\mathfrak{a}^2 = 0$), “space-like" ($\mathfrak{a}^2 < 0$) and “time-like" ($\mathfrak{a}^2 > 0$) respectively.

Case-1 ($\mathfrak{a}^2 = 0$)

It follows quite trivially from (4.75) that for $\mathfrak{a}^2 = 0$ we find no NC effect in the dispersion relation as $M = m$.

[7] As stated in [52, 60], the manifold may not be torsion-free and the metricity criterion may no longer hold. It might not fit into any pseudo-Riemannian manifold. However, all of these topics go outside the scope of this thesis since they call for a study of multi-particle systems and the principles governing the composition of individual particle momenta. Any violation of the associativity and commutativity of the sum of linear momenta is particularly significant.

Case-2 ($\mathfrak{a}^2 < 0$)

In this case (4.75) can be simplified to

$$M = \frac{1}{\sqrt{-\mathfrak{a}^2}}\Big[\sin^{-1}(m\sqrt{-\mathfrak{a}^2})\Big] \tag{4.76}$$

Taylor series expansion around the commutative limit $\mathfrak{a} \to 0$, is given by

$$M = \frac{1}{\sqrt{-\mathfrak{a}^2}}\Big[\lambda + \frac{\lambda^2}{6} + \frac{3\lambda^4}{40} + ...\Big], \qquad \text{for } \lambda = m\sqrt{-\mathfrak{a}^2} < 1 \tag{4.77}$$

Since $\sin^{-1}\lambda$ for $\lambda > 1$ is undefined this naturally puts an upper bound on m as $m < \frac{1}{\sqrt{-\mathfrak{a}^2}}$. The corresponding bound for M is given by $M < \frac{\pi}{2\sqrt{-\mathfrak{a}^2}}$ (see Fig- 4.2).

Case-3 ($\mathfrak{a}^2 > 0$)

In this case the integral (4.75) simplifies to

$$M = \frac{1}{\mathfrak{a}}\sinh^{-1}(\mathfrak{a}m); \qquad \mathfrak{a} = \sqrt{\mathfrak{a}^2} \tag{4.78}$$

The aforementioned equation makes it evident that neither m nor M are strictly speaking bounded [see Fig. 4.2]. There is, however, a subtle point to note. To illustrate this point, it will be of help to remember that for given suitable ranges of the dimensionless parameter $\xi := \mathfrak{a}m$, $\sinh^{-1}(\xi)$ may be Taylor expanded as

$$\sinh^{-1}\xi = \begin{cases} \xi - \frac{\xi^3}{6} + \frac{3\xi^5}{40} - \vartheta(\xi^7) + ... & \text{for } |\xi| < 1 \\ \pm\Big[ln|2\xi| + \frac{1}{4\xi^2} - \frac{3}{32\xi^4} + \vartheta(\xi^{-6}) - ...\Big] & \text{for } \pm\xi \geq 1 \end{cases} \tag{4.79}$$

A smooth commutative limit, as $\xi \to 0$, requires that we consider only the case where $|\xi| < 1$ in (4.79). The other case, where $\xi \geq 1$, clearly does not allow for the desired commutative limit. Therefore, it implies that $\xi = 1$ serves as a critical point. For $\xi < 1$, one can think of constructing an appropriate power series expansion, similar to perturbation theory, to obtain the corresponding noncommutative expression. On the other hand, the regime $\xi > 1$ (shown as the dashed line in Fig. 4.2) cannot be reached from the commutative end by any form of power series expansion. This regime ($m > \frac{1}{\mathfrak{a}}$ or equivalently $M > \frac{1}{\mathfrak{a}}\sinh^{-1}(1) \simeq \frac{0.89}{\mathfrak{a}}$) therefore exhibits some "non-perturbative" features. This is reminiscent of the binomial expansion of Einstein's dispersion relation $E = \sqrt{\vec{P}^2 + m^2}$, which has two regimes: $|\vec{P}| << m$ and $|\vec{P}| >> m$. In the former case, we recover the usual expression for the kinetic energy of a non-relativistic particle: $E = \frac{|\vec{P}|^2}{2m}$ (up to the additive rest mass energy), while in the latter case, we obtain the ultra-relativistic regime $E \sim |\vec{P}|$, where the particle is effectively massless. In Einstein's special relativity, the critical point is given by $\frac{|\vec{P}|}{m} = 1$, which corresponds to the situation described above. The non-relativistic limit in special relativity corresponds to the commutative limit in the previous case, while the ultra-relativistic limit corresponds to the aforementioned "non-perturbative" regime. Just like in the previous case, one cannot obtain the dispersion relation in the ultra-relativistic domain from that of the non-relativistic domain through any form of power series expansion.

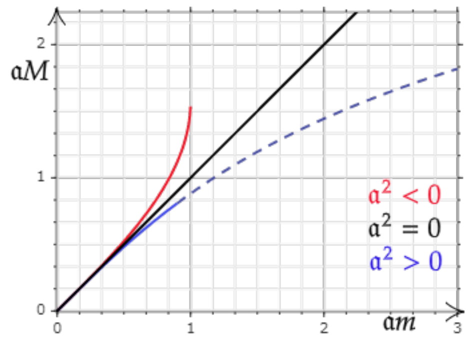

Figure 4.2: Plot of $\mathfrak{a}M$ vs. $\mathfrak{a}m$ where $\mathfrak{a} := \sqrt{\pm\mathfrak{a}^2}$ ($\pm$ for time/space like $\mathfrak{a}^\mu$)

In this context, it is important to recall that the fundamental obstacle in quantizing gravity lies in the presence of a dimensionful coupling constant G, where $\frac{1}{\sqrt{G}}$ corresponds to the Planck mass/energy in natural units. Consequently, any interaction involving quantum gravity necessitates a power series expansion in $\frac{E}{m_P}$, and for $E > m_P$, the sum of this expansion diverges, rendering the theory non-renormalizable. This indicates that our classical analysis of single particle kinematics is likely to break down in the regime $\xi > 1$ (equivalently $m > \frac{1}{\mathfrak{a}}$) for "time-like" $\mathfrak{a}^\mu$.

Finally, it is worth mentioning that the mass M introduced in (4.75) can indeed be identified as a renormalized mass in the spirit of Quantum Field Theory. It can be regarded as the observed mass, in contrast to the bare mass m. To better grasp this concept, we can treat $\mathcal{P}$ as a genuine pseudo-Riemannian manifold, where $D^2(0, P)$ (as given in (4.75)) can be expressed in the following form:

$$D^2(0, P) = (\Pi^0)^2 - (\vec{\Pi})^2 = \eta_{ab}\Pi^a\Pi^b = M^2 \tag{4.80}$$

Here, Π^a's represent the Riemann normal coordinates defined on the tangent plane $T_0(\mathcal{P})$ at the origin $P^\mu = 0$ of the momentum space $\mathcal{P}$. These coordinates are related to the P^μ's through the well-known exponential map [50]. Consequently, one can establish a correspondence between a convex neighborhood $\mathcal{N}_0$ of $T_0(\mathcal{P})$ and another convex neighborhood $\mathcal{N}$ of $\mathcal{P}$ in proximity to the origin. This correspondence allows us to express the following invertible relations for 'space-like', 'null', and 'time-like' $\mathfrak{a}$, respectively

$$\Pi^a = \begin{cases} \frac{P^a}{m\sqrt{-\mathfrak{a}^2}} \sin^{-1}(m\sqrt{-\mathfrak{a}^2}) & \text{for} \quad \mathfrak{a}^2 < 0 \\ P^a & \text{for} \quad \mathfrak{a}^2 = 0 \\ \frac{P^a}{m\sqrt{\mathfrak{a}^2}} \sinh^{-1}(m\sqrt{\mathfrak{a}^2}) & \text{for} \quad \mathfrak{a}^2 > 0 \end{cases} \tag{4.81}$$

where $n^a = \frac{P^a}{m}$; $n^a \in T_0(\mathcal{P})$ is a unit time-like vector tangent to the geodesic at the origin: $n^a n_a = \eta_{ab}n^a n^b = 1$. It can be parameterized as $n^a = (\cosh\psi, \sinh\psi\sin\theta\cos\phi,$ $\sinh\psi\sin\theta\sin\phi, \sinh\psi\cos\theta)$ in terms of the polar coordinates (ψ, θ, ϕ) and can be used to parameterize the space-like 3D hyperboloid $n^a n_a = 1$ ($\mu = 1$ hyperboloid in Fig. 4.1). The momentum space $\mathcal{P}$ can be parameterized in the vicinity of the origin using either a 'polar coordinate' system $(\mu, \psi, \theta, \phi)$ with μ as the radial coordinate (as depicted in Fig. 4.1), or by the original P^α variables themselves (which incidentally coincide with the P^a coordinates used here), or by any other transformed coor-

dinates P'^{μ} obtained through diffeomorphism transformations of P^{μ}, i.e., $P'^{\mu} = P'^{\mu}(P^{\nu})$.
It is worth noting that both the matrices E (as given in (4.59)) and E^{-1} (as given in (4.60)) reduce to identity matrices at the origin. Additionally, without explicit knowledge of the geodesic equation, we cannot construct Riemann normal coordinates at any point along the geodesic away from the origin. However, the Π^{a} coordinates (as defined in (4.67)) hold a special significance, as they can be identified with the components of the renormalized and observed 4-momentum. This interpretation gains further support from the fact that the flat metric $\eta_{\mu\nu}$ can only be recovered from $g_{\mu\nu}(P)$ (as given in (3.22)) in the limit $P^{\mu} \to 0$, akin to the situation where space-time is probed by a soft photon.

Finally, it is important to note that all known elementary particles satisfy this type of bound, and heavier particles can be regarded as composite in nature. Consequently, the total energy and momentum of such composite systems can no longer be obtained by simply adding the momenta of the constituent particles in a general curved momentum space [136]. In general, the total energy and momentum will be less than the sum.

4.2.2 Deformed Lorentz generators

In order to characterize the deformed structures of the Lorentz generators $M_{\mu\nu}$, we consider the kinetic term $Kd\tau$ of the Lagrangian one-form (as given in (4.59)). Specifically, we can identify commutative coordinates q^{μ} as,

$$q^{\mu} = X^{b}E(P)_{b}{}^{\mu} = \phi^{-1}(P)\Big[X^{\mu} + \frac{(\mathfrak{a}.X)P^{\mu}}{\phi(P) - \mathfrak{a}.P}\Big]; \quad p_{\mu} = P_{\mu} \tag{4.82}$$

so that (4.59) can be expressed simply as $Kd\tau = -q^{\mu}dP_{\mu}$, where these new phase space variables satisfy the usual i.e. undeformed phase space algebra, as one can easily check using (4.46)

$$\{q^{\mu}, p_{\nu}\}_{D.B} = \delta^{\mu}{}_{\nu}, \qquad \{q^{\mu}, q^{\nu}\}_{D.B} = 0 \tag{4.83}$$

The pair (q^{μ}, p_{ν}) can therefore be identified as commutative variables. One can check that the inverse transformation of (4.82) is simply given by,

$$X^{a} = (E^{-1}(p))^{a}{}_{\mu}q^{\mu} = q^{a}\phi(p) - (\mathfrak{a}.q)p^{a} \tag{4.84}$$

The existence of the momentum dependent matrix E^{-1} connecting noncommutative and commutative coordinates appears to be a pretty common characteristic in many forms of NC space-times, such as the Bopp shift (2.20) occuring in Moyal spaces [39] and Snyder space-time [137]. It is important to keep in mind that this mapping is solely provided at the classical level and is a *non-canonical* transformation. Therefore, unlike the original X_{μ}, these mathematically specified "position-like" coordinates q_{μ} (4.82) cannot be treated as physical position variables. However, using this coordinate mapping, we can pinpoint the source of distortion in the structure of Lorentz generators, generating the deformed Lorentz transformation (4.25). To illustrate this, take notice of the fact that the standard definition of Lorentz generators of the canonical coordinates q and p fulfilling (4.83) is provided by

$$M_{\mu\nu} = q_{\mu}p_{\nu} - q_{\nu}p_{\mu} \tag{4.85}$$

satisfying all the commutators given in (4.3). Using the above transformation (4.82) enables us to recast it in terms of non-commutative coordinates X^{μ}, we get

$$M_{\mu\nu} = \phi^{-1}(P)(X_{\mu}P_{\nu} - X_{\nu}P_{\mu}) \tag{4.86}$$

By employing (4.86), one can verify that the classical counterparts of (4.3) and (4.21,4.22) are simultaneously satisfied at the level of the classical bracket. It is worth noting that the translation generator P, as we have previously observed, remains undeformed. In the subsequent discussion, we will further confirm, using Noether's approach based on the invariance of the Lagrangian (4.56), that the deformed Lorentz generator indeed possesses the same structure as depicted in (4.86).

4.2.3 Invariance of L under deformed symmetries and Nöther generators

Here, we demonstrate the invariance of the Lagrangian (4.56) under the deformed translation and Lorentz transformation, which correspond to the classical counterparts of (4.25). This implies that the Lagrangian of a relativistic free particle on κ-Minkowski space preserves the same symmetry, namely the deformed Poincaré symmetry, as that of the underlying spacetime itself.
Under deformed translation, the infinitesimal transformations of X and P in the classical context can be expressed as follows:

$$\delta X^{\mu} = -\epsilon^{\alpha}\{P_{\alpha}, X^{\mu}\}_{D.B} = \epsilon^{\mu}\phi(P) - (\epsilon.\mathfrak{a})P^{\mu}; \qquad \delta P_{\mu} = 0 \tag{4.87}$$

where ϵ is the infinitesimal translation parameter. Using (4.87), we can check that the infinitesimal variation of the Lagrangian (4.56) is given by the following quasi-invariant form:

$$\delta L = -\frac{d}{d\tau}(\epsilon.P); \tag{4.88}$$

so that under deformed translations the action remains invariant. We can also derive the translation generator using Noether's prescription, which establishes a connection between the algebraic approach of obtaining generators and the dynamical method presented here.
To demonstrate this, consider that if the variation of a Lagrangian under a particular symmetry is expressed as a total time derivative term, such as $\delta L = \frac{dF}{d\tau}$, then the generator G of the symmetry transformation is given by:

$$G = \Pi^{\xi}_{\mu}\delta\xi^{\mu} - F \tag{4.89}$$

where ξ denotes the configuration space variables. So in our case with the extended configuration space, the translation generator is given by

$$G^{T} = \Pi^{X}_{\mu}\delta X^{\mu} + \Pi^{P}_{\mu}\delta P^{\mu} + (\epsilon.P) = \epsilon.P \tag{4.90}$$

This expression represents the contracted form of the generator with the corresponding parameter ϵ^{μ}, and we can therefore identify the generator as P_{μ} itself. It is important to note that in this analysis, we have not introduced any intermediate translation generator denoted as "∂_{μ}" to represent P_{μ} as done in references [125, 128]. Unlike ∂_{μ}, which possesses an enveloping algebra valued transformation, P_{μ} transforms as a four-vector under Lorentz transformations. We refer to these P_{μ}'s as our *physical momentum space* in contrast to the literature mentioned above.

On the other hand, under deformed Lorentz transformations, the infinitesimal transformations of

X and P can be expressed as follows:

$$\delta X^\mu = \frac{\omega^{\alpha\beta}}{2}\{M_{\alpha\beta}, X^\mu\}_{D.B} = \omega^\mu{}_\alpha X^\alpha + \omega^{\alpha\beta}\mathfrak{a}_\alpha M_\beta{}^\mu$$
$$\delta P_\mu = \frac{\omega^{\alpha\beta}}{2}\{M_{\alpha\beta}, P_\mu\}_{D.B} = \omega_\mu{}^\alpha P_\alpha \tag{4.91}$$

With this we get a complete invariance of the Lagrangian: $\delta L = 0$. The contracted form of the Lorentz generator can be identified in the same way as given in (4.89):

$$G^L = \phi^{-1}(P)\omega_{\mu\nu}X^\mu P^\nu = \frac{1}{2}\omega^{\mu\nu}M_{\mu\nu} \tag{4.92}$$

which exactly matches with the definition (4.86), as mentioned in the previous section.

4.2.4 Finite Lorentz transformation

The non-canonical transformations used in section-4.2.2, (4.82,4.84), can formally be used to define a finite Lorentz transformation. The finite Lorentz transformation in the commutative variables are given by

$$q^\mu \;\to\; q'^\mu = \Lambda^\mu{}_\nu q^\nu; \qquad p_\mu \;\to\; p'_\mu = \Lambda_\mu{}^\nu p_\nu; \quad \Lambda^T\eta\Lambda = \eta \tag{4.93}$$

where $\Lambda \in SO(1,3)$. This will induce the following deformed transformation $\tilde{\Lambda}$ in the NC coordinates X's as

$$X^\mu \to X'^\mu = \tilde{\Lambda}^\mu{}_\nu(\mathfrak{a}, P)X^\nu \tag{4.94}$$

where

$$\tilde{\Lambda}^\mu{}_\nu(\mathfrak{a}, P) = \phi^{-1}(P)\Big[\phi(\Lambda P)\Lambda^\mu{}_\nu + (\Lambda P)^\mu(\mathfrak{a}_\nu - (\mathfrak{a}.\Lambda)_\nu)\Big]$$
$$= \Lambda^\mu{}_\nu + \phi^{-1}(P)\Big[(\mathfrak{a}.\Lambda P)\Lambda^\mu{}_\nu + (\Lambda P)^\mu\mathfrak{a}_\nu - (\mathfrak{a}.P)\Lambda^\mu{}_\nu - (\Lambda P)^\mu\mathfrak{a}_\rho\Lambda^\rho{}_\nu\Big] \tag{4.95}$$

while in the momentum sector the finite transformation remains undeformed.

$$P_\mu \to P'_\mu = \Lambda_\mu{}^\nu P_\nu \tag{4.96}$$

The dependence of $\tilde{\Lambda}$ on P_μ and the deformation parameters $\mathfrak{a}^\mu$ reveals that $\tilde{\Lambda}^\mu{}_{,\nu}$ does not close under ordinary multiplication, in contrast to the undeformed Λ's. This observation suggests that the infinitesimal Lorentz transformation cannot be lifted to a finite one in this scenario.

Now considering an infinitesimal Lorentz transformation $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ for commutative space-time, the corresponding $\tilde{\Lambda}^\mu{}_\nu$ for the non-commutative space-time takes the following form

$$\tilde{\Lambda}^\mu{}_\nu(\mathfrak{a}, P) = \delta^\mu{}_\nu + \tilde{\omega}^\mu{}_\nu(\omega, \mathfrak{a}, P); \qquad \tilde{\omega}^\mu{}_\nu = \omega^\mu{}_\nu - \phi^{-1}(\omega_{\sigma\beta}\delta^\mu{}_\nu P^\beta - \omega_{\sigma\nu}P^\mu)\mathfrak{a}^\sigma, \tag{4.97}$$

so that, X^μ's transforms as

$$X^\mu \to X'^\mu = X^\mu + \delta X^\mu = \tilde{\Lambda}^\mu{}_\nu(\mathfrak{a}, P)X^\nu = (\delta^\mu{}_\nu + \tilde{\omega}^\mu{}_\nu)X^\nu \tag{4.98}$$

Interestingly, this expression for δX^μ can be further simplified to reproduce (4.25) at the classical level: $\delta X^\mu = \frac{1}{2}\omega^{\alpha\beta}\{M_{\alpha\beta}, X^\mu\}$. Once again, it is worth noting that the infinitesimal parameter $\tilde{\omega}^\mu{}_{,\nu}$ depends not only on the undeformed parameter ω, but also on $\mathfrak{a}$ and P.

Furthermore, it is important to observe that the non-vectorial transformation properties of X^μ do not yield a Lorentz invariant quantity $X^\mu X_\mu = X^2$. Instead, the Lorentz invariant quantity $\mathcal{I}$ can be defined using the commutative coordinates q^μ as follows:

$$\mathcal{I} = \eta_{\mu\nu}q^\mu q^\nu = \phi^{-2}(P)\Big[X^\mu X_\mu + \frac{2(\mathfrak{a}.X)(X.P)}{\phi - \mathfrak{a}.P} + \frac{(\mathfrak{a}.X)P^2}{(\phi - \mathfrak{a}.P)^2}\Big] \tag{4.99}$$

This observation indicates that the Lorentz-invariant space-time interval naturally gives way to a different kind of invariant interval that involves the entire phase space variables. In a sense, the entire cotangent bundle defined on the space-time manifold plays a fundamental role in this scenario. This stands in contrast to the deformed space-time interval obtained in (46) of reference [138] for the case where $a^0 = \frac{1}{\kappa}$ and $\vec{a} = 0$. The distinction arises from the deformed structure of their Poincaré algebra itself, which is different from the one in our case.

4.3 Chapter summary

We summarize our findings as follows. In this study, we adopted a bottom-up approach to investigate particle dynamics in generalized κ-Minkowski spaces. The Poincaré symmetry of κ-Minkowski space-time, which remains invariant under the action of a relativistic free particle, plays a fundamental role in understanding the geometry of momentum space and deriving a deformed dispersion relation.
Throughout the analysis, we maintained a fully covariant framework and initially examined the symmetry aspects of κ-Minkowski space-time. By studying the consistency of Jacobi identities, we obtained a set of deformed Poincaré generators while keeping the Poincaré algebra unchanged. It is worth noting that the resulting Heisenberg algebra (4.22) is non-standard, and this deformation is shown to have implications in the theory, as evidenced by the identification of tetrads in curved momentum space $\mathcal{P}$. We then constructed an explicit dynamical model that respects the associated symmetries by formulating an appropriate first-order Lagrangian (4.56) for a massive and spinless free relativistic particle in deformed κ-Minkowski space. Our approach demonstrates the effectiveness of Dirac's constraint analysis or the symplectic method. Initially, the Hamiltonian was considered as an undetermined function of the Poincaré Casimir (P^2), as this generalization does not affect the Dirac/symplectic structure of the theory. However, we discovered that this generalization has a significant impact on the energy dispersion relation, particularly through the observation of the first term in our Lagrangian (4.56). We identified the presence of a tetrad factor $E(P)_\alpha{}^\mu$ (4.59) in momentum space $\mathcal{P}$, which connects the global momentum variables P^μ with the local momentum variables P^a. This, in turn, leads to the emergence of non-holonomic bases e^a (4.21) in the cotangent space of the curved momentum manifold $\mathcal{P}$. This supports existing views in the literature regarding the curved geometry of momentum space associated with κ-Minkowski-type deformations of the space-time manifold. The deformation in the Heisenberg algebra (4.22), resulting from the consistency of the Jacobi identities, which is a consequence of the stability of the κ algebra under different symmetry operations, is believed to be the key factor responsible for the deformations in the spectrum even for a free particle in κ-Minkowski spacetime. Additionally, we provided a mapping between the

κ-Minkowski position coordinates and the usual commutative coordinates, reminiscent of the well-known Bopp transformations (2.20) [39] that connect noncommutative and commutative coordinates in the case of the Moyal plane. Utilizing this transformation, we explicitly computed the deformed Lorentz generators, which were subsequently verified through Noether's analysis on the Lagrangian.

The present analysis unveils a novel dispersion relation for a single particle, which has not been previously reported. This newly derived dispersion relation holds the potential to significantly impact some astrophysical objects, particularly its influence on the equation of state. By revisiting Chandrasekhar's analysis with this modified equation of state, it may alter the mass limit imposed on white dwarfs [139]. Furthermore, we establish a relationship between the mass M (referred to as the renormalized mass) of a particle in κ-Minkowski space-time and its mass m (known as the bare mass) in commutative space-time. In our investigation, we examine three cases. For a light-like deformation parameter, there is no deformation in the bare mass. Conversely, for a space-like deformation parameter, the bare mass exhibits an upper bound of $m < \frac{1}{\mathfrak{a}}$, along with a constrained renormalized mass $M < \frac{\pi}{2\mathfrak{a}}$. In the case of a time-like deformation parameter, M becomes a monotonically increasing function of m. While there is no upper bound on the mass m or M in this scenario as such, the theory indicates the existence of a mass scale, anticipated to be on the order of the Planck mass. Beyond this scale, the growth of M becomes virtually independent of that of m, resulting in the renormalized mass sort of plateauing within a narrow range. This observation raises intriguing questions about the nature of space-time itself and the measurements made in this regime of quantum gravity, ultimately relating to the concept of relative locality [52]. An important outcome of our analysis is that the momentum space $\mathcal{P}$ in our case cannot be identified as a group manifold, contrary to the findings of [60]. The momentum manifold $\mathcal{P}$ now encompasses four distinct parameters $\mathfrak{a}^\mu$, each with its own significance. In situations where these parameters or scales significantly differ, the manifold $\mathcal{P}$ cannot correspond to a maximally symmetric space. Even in cases where $\mathfrak{a}^\mu$ is purely time-like, meaning $\mathfrak{a}^0 \neq 0$ and $\vec{\mathfrak{a}} = 0$, the recovery of the group manifold AN(3) from [60] remains unattainable. This discrepancy likely stems from the Lorentz covariant transformation properties of the P^μ's in our analysis, which contrast with those of [60]. Consequently, the composition of individual momenta in a multi-particle system to obtain the total momentum becomes challenging [136]. Exploring the construction of multi-particle actions in the presence of interactions, such as simple collisions, and studying the corresponding Hopf algebra symmetry holds great interest. Moreover, through the dynamical analysis of deformed Poincaré symmetry, we observe that the action of finite Lorentz transformations on κ Minkowski space-time yields an invariant quantity (4.99) that depends on all the phase space variables. This quantity can be interpreted as a generalization of the conventional space-time interval. This observation underscores the notion that the geometry of the entire cotangent bundle emerges as the fundamental object in this scenario. The introduction of a mass scale acts as a unifying factor, integrating the coordinate space and momentum space, similar to how the introduction of a universal speed unifies space and time in special relativity [52]. Consequently, this represents a paradigm shift from the conventional geometric standpoint, perhaps indicating a more fundamental role played by phase space, rather than space-time itself and may have a connection with Finsler geometry [140].

Furthermore, our proposed geometric implications of deformed space-time symmetries hold the promise of resolving various issues. For instance, an essential investigation would involve exploring quantum gravity (QG) corrections to the world line path-integral formulation of effective quantum field theory [141]. This formulation is grounded in the classical action governing a relativistic charged particle in the presence of a background gauge potential.

Additionally, delving into the case of a relativistic spinning particle in a κ-Minkowski background presents an intriguing opportunity. In this scenario, the geometry of space departs from the Riemannian framework due to the induction of torsion resulting from spinning particles. This avenue opens up captivating questions that await further exploration.

Part- B

Chapter 5
Spectral distance on Lorentzian Moyal plane

As was discussed in the chapter-1, Alain Connes formulated Non-commutative geometry (NCG) [73] and applied it to fully describe the particle content-the gauge fields unified with Higgs fields and fermionic fields of the standard model along with the gravitational action albeit at the classical level [142,143]. However, his formulation primarily focused on spaces with Euclidean signature, specifically Riemannian manifolds. This aspect has posed a challenge for further development and the fact that it is not in alignment with the realistic nature of our space-time, modeled as a differentiable manifold with Lorentzian signature. Various attempts have been made to address this issue, such as the "Wick rotation" method [143] and others [65]. Recent activities in this direction have shown a lack of consensus in the literature regarding the axiomatic formulation of spectral triplet, appropriate for Lorentzian signature [144–149,151]. Despite the ongoing debate, our study follows the work of Franco et al. [151,152] in a preliminary attempt to compute the spectral distance between a pair of time-like separated events represented by pure states of a C^*-algebra. We focus in this chapter, on the Lorentzian Moyal plane and demonstrate that the axiomatic formulations proposed by Franco et al. adequately serve our purpose. This work can be seen as a sequel to the previous research on the Euclidean Moyal plane [42], where authors also utilized the novel Hilbert-Schmidt operator formulation of Non-commutative (NC) quantum mechanics [21], as reviewed in chapter-2. To the best of our knowledge, the presented computation of distance in the Lorentzian Moyal plane represents the first attempt in this direction.

The chapter is organized as follows: In section-5.1, we first revisit the computation of spectral distance in Euclidean Moyal plane using the spectral triple given in section-2.2 of chapter-2. We utilize the Bargmann-Fock coherent state basis to represent states in the Hilbert space and employ the functional differentiation approach to extremize various functionals that naturally arise in deriving the spectral distance between a pair of pure states on the Euclidean Moyal plane. In section-5.2, we provide a brief outline of the construction of the Lorentzian spectral triple and discuss the emergence of the algebraic ball condition as a result of causality, following the work of Franco et al. [151,152]. The subsequent section focuses on the explicit derivation of the Dirac operator and the ball condition for two-dimensional Lorentzian commutative and non-commutative manifolds with (-,+) signature. We then calculate distances using the functional derivative approach for these respective manifolds. Additionally, we discuss the Poincaré invariance of the distance and the non-invariance of the "vacuum" under Lorentz boost. We conclude with some comments and future direction of this work in chapter-5.4.

5.1 Computation of spectral distance on Euclidean Moyal plane

In this section, we will provide a concise summary of the computation of spectral distance in the Euclidean Moyal plane, as previously presented in [42]. However, in contrast to [42], we will utilize the Bargmann-Fock coherent states to represent state vectors in the Hilbert space, facilitating a more transparent computation and enhancing readability for the potential reader. This approach will establish the foundational framework and introduce the necessary notation, which can then be adapted for the Lorentzian case. Remarkably, we will discover several points of connection between these two cases, highlighting both their similarities and differences.
As discussed in chapter-2, the algebra governing the operator-valued coordinates of the Moyal plane is given by:

$$[\hat{x}_1, \hat{x}_2] = i\theta; \;\; \theta > 0 \tag{5.1}$$

The form of the commutator algebra (5.1) does not determine the signature of the underlying commutative space; it can be either Euclidean or Lorentzian. This is evident from the fact that the NC algebra (5.1) is invariant under both SO(2) and SO(1,1) transformations. However, Connes' formulation of NCG [76], requires the spin manifold to be Euclidean. Therefore, initial studies on computing spectral distances between pure states in the Moyal plane [157] and fuzzy sphere [150] were limited to Euclidean signatures, as the corresponding commutative limits yield manifolds with the associated Euclidean metrics. Notably, in [42], computations were conducted at an operatorial level using the Hilbert-Schmidt operator, which avoids the use of star product and any resulting ambiguities [38]. In the following, we will perform the same computation using a simpler approach, which will serve as a foundation for computing Lorentzian spectral distance.

The spectral distance on Moyal plane is computed by employing the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ (See section-2.2 for details) where

$$\mathcal{A} = \mathcal{H}_q, \;\; \mathcal{H} = \mathbb{C}^2 \otimes \mathcal{H}_c, \;\; \mathcal{D} = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & \hat{b}^\dagger \\ \hat{b} & 0 \end{pmatrix}. \tag{5.2}$$

The space $\mathcal{H}_q$ corresponds to the set of Hilbert-Schmidt (HS) operators (2.6) operating on the "configuration space" $\mathcal{H}_c$ (2.4). Both $\mathcal{H}_q$ and $\mathcal{H}_c$ are Hilbert spaces, as defined in the chapter 2. The Hilbert space $\mathcal{H}_q$, with generic forms such as (2.9), possesses the structure of an algebra. This allows us to identify the algebra $\mathcal{A}$ with $\mathcal{H}_q$ itself, i.e., $\mathcal{A} = \mathcal{H}_q$. The algebra acts on $\mathbb{C}^2 \otimes \mathcal{H}_c$ through the diagonal representation $\pi(a) := \text{diag}(a, a)$ from the left, enabling us to view $\mathbb{C}^2 \otimes \mathcal{H}_c$ as the left module of the algebra. Finally, in (5.2), $\mathcal{D}$ represents the Dirac operator, whose construction has been reviewed in chapter-2 following [42].

In this context, there are two choices for states: the normalized Sudarshan-Glauber coherent state given in (2.12) and the states $|n\rangle$ (2.4) as introduced in chapter-2 . Correspondingly, one can introduce pure states $\rho_z := |z\rangle\langle z|$, and the so called "harmonic oscillator" states $\rho_n := |n\rangle\langle n| \in \mathcal{H}_q$ which are linear functionals of unit norm, acting on the algebra $\mathcal{A} = \mathcal{H}_q$ as

$$\rho_z(\hat{a}) = \text{tr}_{\mathcal{H}_c}(\rho_z \hat{a}) = (\rho_z, \hat{a}) = \langle z|\hat{a}|z\rangle \;; \rho_n(\hat{a}) = tr_{\mathcal{H}_c}(\rho_n \hat{a}) = \langle n|\hat{a}|n\rangle \tag{5.3}$$

These pure states ρ_z and ρ_n can also be interpreted as density matrices when viewed from $\mathcal{H}_c$. Following the approach of Gel'fand and Naimark, we can associate ρ_z, in particular, with the point having the complex coordinate z in the Argand diagram, where the latter is considered as a smeared Moyal plane. Here we shall find the spectral distance between a generic pair of states ρ_z and ρ_w using Connes' distance formula, as given by (2.25).

Given the structure of $\mathcal{D}$ in (5.2), the ball condition (2.26) can be expressed equivalently and more concisely as follows:

$$\mathcal{B} = \left\{ \hat{a} \,:\, \|[\hat{b}, \hat{a}]\|_{op} = \|[\hat{b}^\dagger, \hat{a}]\|_{op} \leq \sqrt{\frac{\theta}{2}} \right\} \tag{5.4}$$

As has been noted in [42] that the optimal algebra element $a_s \in \mathcal{A}$ for which the supremum is attained in (2.25) , yielding the distance i.e.

$$d(\rho_z, \rho_w) = |\rho_z(\hat{a}_s) - \rho_w(\hat{a}_s)| \tag{5.5}$$

must also saturate the ball condition (5.4) i.e. it should satisfy

$$\|[\hat{b}, \hat{a}_s]\|_{op} = \|[\hat{b}^\dagger, \hat{a}_s]\|_{op} = \sqrt{\frac{\theta}{2}} \tag{5.6}$$

Further following [160], we know that the search of such an optimal algebra element can be restricted to hermitian algebra elements only i.e. we require $\hat{a}_s^\dagger = \hat{a}_s$. To that end, we will consider the simplified ball condition (5.4) and focus on computing $\|[\hat{b}, \hat{a}]\|_{op}^2$. By utilizing the Bargmann-Fock basis (F.1) and the corresponding resolution of identity (F.2) (refer to Appendix-F), we obtain the following expression:

$$\|[\hat{b}, \hat{a}]\|_{op}^2 = \sup_{\|\psi\|=1} \langle\psi|[\hat{b}, \hat{a}]^\dagger [\hat{b}, \hat{a}]|\psi\rangle = \sup_{\|\psi\|=1} \int d\mu(z, \bar{z}) d\mu(w, \bar{w}) \psi^*(z)\psi(\bar{w})\langle\bar{z}|[\hat{b}, \hat{a}]^\dagger [\hat{b}, \hat{a}]|w\rangle \tag{5.7}$$

One approach to compute the supremum is to utilize the method of Lagrange's undetermined multiplier to extremize the right-hand side of the above equation, subjected to the constraint $\langle\psi|\psi\rangle = 1$. Hence, we aim to extremize the following functional:

$$\mathbf{B}[\psi^*(z), \psi(\bar{z}); \lambda] := \int d\mu(z, \bar{z}) d\mu(w, \bar{w}) \psi^*(z)\psi(\bar{w})\langle\bar{z}|[\hat{b}, \hat{a}]^\dagger [\hat{b}, \hat{a}]|w\rangle - \lambda \left[\int d\mu(\bar{z}, z)\psi^*(z)\psi(\bar{z}) - 1 \right] \tag{5.8}$$

with λ being a real valued Lagrange's multiplier enforcing the constraint $\langle\psi|\psi\rangle = 1$. By varying $\mathbf{B}[\psi^*(z), \psi(\bar{z}); \lambda]$ with respect to $\psi(\bar{z}), \psi^*(z)$ we get the following pair of equivalent equations,

$$\begin{aligned} \langle\psi|[\hat{b}, \hat{a}]^\dagger [\hat{b}, \hat{a}]|z\rangle &= \lambda\langle\psi|z\rangle \\ \langle\bar{z}|[\hat{b}, \hat{a}]^\dagger [\hat{b}, \hat{a}]|\psi\rangle &= \lambda\langle\bar{z}|\psi\rangle. \end{aligned} \tag{5.9}$$

In carrying out this functional differentiation we have used the basic relations like (F.5) and (F.6) (see Appendix-F). Now since (5.9) is valid for any arbitrary $|\psi\rangle$ and $|z\rangle$, the matrix elements of the operator $[\hat{b}, \hat{a}]^\dagger [\hat{b}, \hat{a}]$ between any pair of states $|\psi\rangle$ and $|\phi\rangle$ must satisfy $\langle\psi|[\hat{b}, \hat{a}]^\dagger [\hat{b}, \hat{a}]|\phi\rangle = \lambda\langle\psi|\phi\rangle;\ \forall\ |\psi\rangle, |\phi\rangle \in$

$\mathcal{H}_c$. It thus follows that we must have the following operator identity :

$$[\hat{b}, \hat{a}]^\dagger [\hat{b}, \hat{a}] = \lambda \tag{5.10}$$

The structure of this equation suggests that $\hat{a}$ may only have a linear relationship with $\hat{b}$ and $\hat{b}^\dagger$, yielding a real number on the right-hand side, as evident from the general form of the algebra element $\hat{a}$ in (2.10). However, this is not entirely true. In fact, as shown in [42], there exist finite-dimensional matrix solutions for $\hat{a}$ such that the positive operator $[\hat{b}, \hat{a}]^\dagger [\hat{b}, \hat{a}]$ is a diagonal matrix, but not proportional to identity matrix. Consequently, one can identify states $|\psi_i\rangle \in \mathcal{H}_c$ corresponding to local extrema with associated eigenvalues $\lambda_i \geq 0$, satisfying $[\hat{b}, \hat{a}]^\dagger [\hat{b}, \hat{a}]|\psi_i\rangle = \lambda_i |\psi_i\rangle$. The maximum eigenvalue $\lambda_{\max}$ can then be determined, and the operator norm $\|[\hat{b}, \hat{a}]\|_{op}$ can be computed as $\sqrt{\lambda_{\max}}$ (see Appendix-G). These finite-dimensional matrix solutions for $\hat{a}$, which result in a distance with $\lambda_{\max} = \frac{\theta}{2}$, are utilized to compute distances between the "harmonic oscillator" states ρ_n and ρ_m, where they serve as an optimal element. In our case, however, we aim to compute the distance between a pair of coherent states ρ_z and ρ_w (2.25). From the definition (2.12) of coherent states, it is evident that we require a non-trivial infinite-dimensional solution for the optimal element $\hat{a}_s$ that yields a distance with a manifest invariance property under ISO(2) [42]. Additionally, we can provide an upper bound for the distance (2.25), which is then shown to be saturated by the optimal element $\hat{a}_s$.

It is useful, at this stage, to briefly recapitulate this derivation. Here, we introduce a one-parameter family of pure states $\rho_{\mu z} := |\mu z\rangle\langle \mu z|$ with $\mu \in [0, 1]$ being a real parameter, interpolating between ρ_0 and ρ_z (we have assumed, without loss of generality, $w = 0$ by invoking the aforementioned ISO(2) invariance). We can then rewrite the expression appearing on the right-hand side of (2.25) as:

$$|\rho_z(\hat{a}) - \rho_0(\hat{a})| = \left| \int_0^1 d\mu \frac{d\rho_{\mu z}(\hat{a})}{d\mu} \right| \leq \int_0^1 d\mu \left| \frac{d\rho_{\mu z}(\hat{a})}{d\mu} \right| = \int_0^1 d\mu \left| \bar{z}\rho_{\mu z}[\hat{b}, \hat{a}] + z\rho_{\mu z}[\hat{b}, \hat{a}]^\dagger \right| \tag{5.11}$$

By applying Cauchy-Schwartz inequality, this inequality can further be simplified as ,

$$|\rho_z(\hat{a}) - \rho_0(\hat{a})| \leq \sqrt{2}|z| \int_0^1 d\mu \sqrt{\left|\rho_{\mu z}[\hat{b}, \hat{a}]\right|^2 + \left|\rho_{\mu z}[\hat{b}, \hat{a}]^\dagger\right|^2} \leq 2|z| \sqrt{\|[\hat{b}, \hat{a}]^\dagger [\hat{b}, \hat{a}]\|_{op}} = 2|z| \ \|[\hat{b}, \hat{a}]\|_{op} \tag{5.12}$$

Finally making use of the ball condition (5.10), we get the desired upper bound on the distance as,

$$d(\rho_z, \rho_0) \leq \sqrt{2\theta}|z| \tag{5.13}$$

Equivalently by re-introducing z and w variables again by invoking ISO(2) symmetry we have,

$$d(\rho_z, \rho_w) \leq \sqrt{2\theta}|z - w| \tag{5.14}$$

We can now take the ansatz for $\hat{a}$ as $\hat{a} = \xi\hat{b} + \bar{\xi}\hat{b}^\dagger$ with $\xi \in \mathbb{C}$ (as the algebra element should be hermitian). We have from (5.10), $|\xi|^2 = \lambda$. To evaluate the value of λ we make use of (5.4), (5.7)and (5.10) to have,

$$\|[\hat{b}, \hat{a}]\|_{op} = \sup_{\|\psi\|=1} \sqrt{\langle\psi|[\hat{b}, \hat{a}]^\dagger [\hat{b}, \hat{a}]|\psi\rangle} = \sqrt{\lambda} = \sqrt{\frac{\theta}{2}} \tag{5.15}$$

So the value of λ for which the ball reaches its supremum value is $\lambda = \frac{\theta}{2}$. So the optimal algebra element $\hat{a}_s$ lies within the one parameter family given by the phase α as,

$$\hat{a}_s \in \left\{ \sqrt{\frac{\theta}{2}}(\hat{b}e^{-i\alpha} + \hat{b}^\dagger e^{i\alpha});\ 0 \leq \alpha < 2\pi \right\} \tag{5.16}$$

where we have taken $\xi = e^{-i\alpha}\sqrt{\frac{\theta}{2}}$. One can further corroborate this by observing that the matrix element of $[\hat{b}, \hat{a}_s]$ with $\hat{a}_s$ given by (5.16), in the Bergman-Fock basis (F.1) is obtained as,

$$_B\langle \bar{z}'|[\hat{b}, \hat{a}_s]|z\rangle_B = \sqrt{\frac{\theta}{2}} e^{i\alpha} e^{\bar{z}' z} \tag{5.17}$$

Here, we observe the appearance of Dirac's delta function in the Bargmann-Fock space: $e^{\bar{z}'z} = \delta(\bar{z}', z)$ (see (F.6) in Appendix-F). This leads to the proportionality of $[\hat{b}, \hat{a}_s]$ to the unit matrix when expressed in this continuous Bargmann-Fock basis, allowing us to directly determine the operator norm. It is important to note that, unlike the finite-dimensional matrix $\hat{a}_s$ (see (G.1) in Appendix-G), the $\hat{a}_s$ in (5.16) yields a unique eigenvalue $\lambda = \frac{\theta}{2}$ for $[\hat{b}, \hat{a}]^\dagger[\hat{b}, \hat{a}]$, which is infinitely degenerate in the eigen-basis provided by the entire continuum set of Bargmann-Fock coherent states $|z\rangle$ (F.1). Importantly, this eigenvalue is independent of z, resulting in translation-invariant distances, in contrast to the distances between harmonic oscillator states. For instance, $d(\rho_{n+1}, \rho_n) \neq d(\rho_{n+2}, \rho_{n+1})$ [42,161]. Furthermore, this identification of λ enables the upper bound on the distance to take the form given by (5.14).
By substituting (5.16) into (2.25), we find that the distance is essentially given by:

$$d(\rho_z, \rho_w) = \sup_{\hat{a} \in \mathcal{A}} |\langle z|\hat{a}_s|z\rangle - \langle w|\hat{a}_s|w\rangle| = \sqrt{\frac{\theta}{2}} \sup_{\alpha} |(z-w)e^{-i\alpha} + (\bar{z} - \bar{w})e^{i\alpha}| \tag{5.18}$$

Note that since ρ_z and ρ_w are normalized pure state density matrices, we need to use the normalized coherent state basis, as defined in (2.12), for distance calculation, instead of the Bargmann-Fock basis[1]. We can now parameterize the complex number using polar decomposition as $z - w = |z - w|e^{i\beta}$ and $\bar{z} - \bar{w} = |z - w|e^{-i\beta}$. It is easy to recognize that the optimal algebra element, for which the supremum value is attained, occurs when $\alpha = \beta$, yielding the desired distance between two arbitrary pure states:

$$d(\rho_z, \rho_w) = \sqrt{2\theta}\ |z - w| \tag{5.19}$$

which precisely is the upper bound (5.14). Therefore $\hat{a}_s$ (5.16) saturates the inequality. The dimensionless complex coordinates z and w can be splitted into real and imaginary parts involving dimensionful coordinates $\vec{x}$ and $\vec{y}$ in analogy with the splitting of $\hat{b}$ in (2.4) as

$$z = \frac{1}{\sqrt{2\theta}}(x_1 + ix_2) \text{ and } w = \frac{1}{\sqrt{2\theta}}(y_1 + iy_2) \tag{5.20}$$

which enables us to rewrite (5.19) using (5.20) as,

$$d(\rho_z, \rho_w) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \tag{5.21}$$

[1]To avoid confusion, we will use the same notation $|z\rangle$ to denote both the normalized coherent state (2.12) and the Bargmann-Fock (unnormalized basis) $|z\rangle_B$ in (F.1). The context should make it clear which one is being used and should not cause any confusion. Specifically, when $|z\rangle$ appears in the density matrix $\rho_z = |z\rangle\langle z|$, it clearly refers to the normalized state, otherwise, it refers to the latter one.

reproducing the usual Euclidean distance having the complete ISO(2) symmetry as in [42,157], with no NC corrections.

The absence of any NC correction, initially observed in [157] and later in [42], is not particularly surprising. The Moyal plane represents a NC deformation of a flat 2D plane, and the family of coherent states ρ_z can be uniquely associated with points in this plane, characterized by the dimensionless complex coordinate z. The distance (5.21) endows the space of states ρ_z, or equivalently the 2D plane, with a flat Euclidean metric, providing a measure of distance between points (x_1, x_2) and (y_1, y_2). These points represent the mean positions of maximally localized $(\Delta x_1 \Delta x_2 = \frac{\theta}{2})$ SO(2)-symmetric coherent states. However, the metric structure mentioned above captures only the distance between pairs of mean positions represented by pure states in the Euclidean sense and does not explicitly depend on the spatial spreads of the coherent states. On the other hand, the so-called harmonic oscillator states $\rho_n := |n\rangle\langle n|$, constructed from the Fock states (2.4), are completely delocalized and can be visualized as a family of concentric circles indexed by the integer n, with quantized radii $r_n = \theta(2n+1)$. The distance between a pair of successive such states (see (G.3) in Appendix-G) exhibits NC deformations. Another example of spectral distance exhibiting NC deformation is the fuzzy sphere S^2_* [164], defined through the commutation relation $[\hat{x}_i, \hat{x}_j] = i\theta\epsilon_{ijk}\hat{x}_k$, which corresponds to the $\mathfrak{su}(2)$ algebra, where θ is the NC deformation parameter (with dimensions of length), in contrast to the one occurring in (5.1) for the Moyal plane. In the case of the fuzzy sphere, the radii are quantized according to the spectrum of the $\mathfrak{su}(2)$ Casimir operator, and the NC deformation arises from both the NC parameter θ and the $\mathfrak{su}(2)$ representation index j. Specifically, the fuzzy sphere with $j = \frac{1}{2}$ representation of the $\mathfrak{su}(2)$ algebra exhibits the maximal deformation in the spectral distance, and the commutative result is recovered only in the limit $j \to \infty$ [42,150].

Before concluding this section, it is important to note that the form of the algebra element given in (5.16) does not belong to the algebra itself, i.e., $a_s \notin \mathcal{A}$. However, these elements can be considered as part of an enlarged algebra known as the multiplier algebra (for more details, see [157,159]). We will not delve further into this topic here, but it is worth mentioning that in our case, the operator product of ρ_z and $\hat{a}_s$ in (5.16) is indeed a Hilbert-Schmidt (HS) operator. This can be inferred from the fact that $\|\rho_z \hat{a}_s\|^2_{HS} = \mathrm{tr}_{\mathcal{H}_c}((\rho_z \hat{a}_s)^\dagger (\rho_z \hat{a}_s)) = \mathrm{tr}_{\mathcal{H}_c}(\hat{a}_s \rho_z \hat{a}_s) < \infty$. This property will recur throughout the rest of the chapter. In fact, in the subsequent section, when we discuss the Lorentzian Moyal plane, we will also need to consider the multiplier algebra. In the Lorentzian case also, as we shall see later that there is no NC deformation in the distance function between time-like separated events in the Lorentzian Moyal plane, provided that the pure states are constructed using coherent states.

5.2 Lorentzian spectral triple

In this section, we aim to utilize a formulation necessary for calculating distances between a pair of time-like separated events in both commutative and NC Lorentzian planes. While a consensus on the axiomatic frameworks for such formulations is yet to be reached, we will follow the approach presented in [151–153], adapting it to our HS-operatorial formulation discussed in the previous section for the Euclidean case. Our goal is to introduce the key tools, such as spectral triples, for both commutative and NC spaces. Additionally, we will include the algebraic version of causality, which is essential for facilitating the computation of distances in the commutative and NC Lorentzian planes, as we will explore in the subsequent section.

To begin with, we provide a concise outline of the fundamental ingredients required to construct a Lorentzian spectral triple. This involves a collection of data denoted by $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{J}, \chi)$, where

- A Hilbert space $\mathcal{H}$
- A non-unital algebra $\mathcal{A}$ along with a suitable and faithful representation $\pi(a)$ on the Hilbert space $\mathcal{H}$. Additionally, we consider a preferred unitization $\tilde{\mathcal{A}}$ of $\mathcal{A}$ equipped with a representation as bounded operators on the Hilbert space.
- A Dirac operator $\mathcal{D}$ taken to be an unbounded operator, such that $[\mathcal{D}, \pi(a)]$ is bounded $\forall\, a \in \tilde{\mathcal{A}}$. Furthermore, $\mathcal{D}$ should have a compact resolvent i.e. $\pi(a)[1 + \frac{1}{2}(D^*D + DD^*)]^{-\frac{1}{2}}$ is a compact operator $\forall a \in \mathcal{A}$. (However this condition need not be fulfilled in this context. In other words, one can forgo this requirement [151], like in the Euclidean case [157], as we are interested to study the metric aspect only.)
- An operator $\mathcal{J}$ called Fundamental symmetry acting on $\mathcal{H}$ satisfying boundedness condition along with $\mathcal{J}^2 = 1,\ \mathcal{J}^* = \mathcal{J},\ [\mathcal{J}, a] = 0,\ \forall a \in \tilde{\mathcal{A}}$, and the Dirac operator fulfills $\mathcal{D}^* = -\mathcal{J}\mathcal{D}\mathcal{J}$.
- For an even Lorentzian spectral triple we take χ to be a grading operator fulfilling the following conditions:
$$\chi^* = \chi;\ \chi^2 = 1;\ \{\chi, \mathcal{J}\} = 0;\ \{\mathcal{D}, \chi\} = 0. \tag{5.22}$$

The operator $\mathcal{J}$ plays a crucial role in capturing the Lorentzian signature of the space. It transforms the Hilbert space into a Krein space, where the positive definite inner product $\langle \cdot, \cdot \rangle$ is transformed to an indefinite inner product $(\cdot, \cdot) = \langle \cdot, \mathcal{J} \cdot \rangle$. This transformation is accomplished by the fundamental symmetry operator $\mathcal{J}$. Consequently, the Dirac operator in the usual positive definite inner product space becomes a non-Hermitian operator. To ensure that the Dirac operator is a skew Krein self-adjoint operator, we impose the condition $\mathcal{D}^* = -\mathcal{J}\mathcal{D}\mathcal{J}$. The detailed construction and implications of this formulation will be discussed in section-5.3.

5.2.1 Commutative Lorentzian spectral triple and distance formula

In this subsection, we consider the example of a globally hyperbolic even-dimensional manifold $\mathcal{M}$ as a commutative Lorentzian spacetime. We describe this spacetime using a commutative spectral triple [151], which is constructed with the following elements:

- Hilbert space $\mathcal{H} = L^2(\mathcal{M}, S)$ of square integrable spinorial sections over $\mathcal{M}$
- Dirac operator $\mathcal{D} = -i\gamma^\mu \nabla_\mu$
- Algebra $\mathcal{A} = C_0^\infty(\mathcal{M})$ of infinitely differentiable smooth real functions. (Ideally, one should introduce the unitized version of the algebra- a fact that we are ignoring for the time being. We shall, however, make some pertinent observations later in the chapter.)
- Fundamental symmetry $\mathcal{J} = i\gamma^0$ where γ^0 is the time- component of Dirac's γ- matrices γ^μ satisfying the fundamental Clifford algebra : $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{I}$ and the flat metric $\eta_{\mu\nu}$ =diag(-,+,+,+,...(n-1)terms)
- Grading operator $\chi = (-i)^{\frac{n}{2}+1}\gamma^0 \cdots \gamma^{n-1}$

For an even-dimensional Lorentzian manifold, i.e., when n is even, the distance between two points is essentially identified with the distance between the associated pure states. These pure states are given by Dirac's delta functionals, which serve as evaluation maps, and can be expressed as follows [151]:

$$d(p,q) = \inf_{a\in\mathcal{B}}\{[\delta_q(a) - \delta_p(a)]^+\} = \inf_{a\in\mathcal{B}}\{[a(q) - a(p)]^+\} \tag{5.23}$$

where $[\alpha]^+ = \max\{0, \alpha\}$, and

$$\mathcal{B} = \{a \in \mathcal{A} : \mathbf{B} := \langle\phi, \mathcal{J}([\mathcal{D}, a] \pm i\chi)\phi\rangle < 0, \;\; \forall\phi \in \mathcal{H}\} \subset \mathcal{A} \tag{5.24}$$

In the next subsection, we will explore how this specific ball condition captures the concept of steep functions as a subset of causal functions in an algebraic framework, drawing inspiration from [151, 155].

5.2.2 Non-commutative Lorentzian spectral triple and distance formula

Now, the spectral triple for non-commutative (NC) Lorentzian space-time, specifically the (1+1) D Lorentzian Moyal plane, can be readily generalized. It consists of the following components:

- a Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathcal{H}_c$
- Non-commutative algebra $\mathcal{A} = \mathcal{H}_q$ acting on $\mathcal{H}$ with a suitable representation π, again taken to be diagonal one i.e. $\pi(a) := \text{diag}(a, a)$.
- Dirac operator $\mathcal{D} = -i\gamma^\mu\nabla_\mu$

where we have already defined $\mathcal{H}_c, \mathcal{H}_q$ for Moyal space-time while discussing about space-time non-commutativity in chapter-3 in (3.28) and (3.30). In the case of the non-commutative (NC) plane, the fundamental symmetry operator and grading operator remain the same as defined in the commutative spectral triple. However, the concept of a point is absent in the NC plane. Therefore, the distance is calculated between two pure states, denoted as ρ_1 and ρ_2, similar to its Euclidean counterpart mentioned in section-5.1. The Lorentzian distance between these two states ρ_1 and ρ_2 is defined as follows:

$$d(\rho_1, \rho_2) = \inf_{a\in\mathcal{B}}\{[\rho_2(a) - \rho_1(a)]^+\} \tag{5.25}$$

where $[\alpha]^+ = \max\{0, \alpha\}$ and the ball $\mathcal{B}$ is defined as

$$\mathcal{B} = \{a \in \mathcal{A} = \mathcal{H}_q \;:\; \mathbf{B} = \langle\phi, \mathcal{J}([\mathcal{D}, \pi(a)] \pm i\chi)\phi\rangle < 0, \forall\phi \in \mathcal{H}\} \subset \mathcal{A} \tag{5.26}$$

Note that this distance formula works only for the even dimensional case, so that an appropriate grading operator χ can be defined. There is also an alternative formula for odd dimensional manifold [151], which is, however, beyond the scope of this thesis.

5.2.3 An algebraic construction of causality

In this section we provide a very brief review following [151,154,155] to show how the causal structure is captured as an algebraic condition in Lorentzian space.

Theorem : A function $a \in C^1(\mathcal{M}, \mathbb{R})$ is causal *iff*

$$\forall\phi \in \mathcal{H}, \;\; \langle\phi, \mathcal{J}[\mathcal{D}, a]\phi\rangle < 0 \tag{5.27}$$

where $\mathcal{J}, \mathcal{D}, \mathcal{H}$ are defined as above. A sketch of the proof for the algebraic version can be outlined as follow.
Recall that the causal property of an absolutely continuous real valued function $a \in C^1(\mathcal{M}, \mathbb{R})$, can be fully characterised by two conditions i.e.

$$\eta(\nabla a, \nabla a) < 0 \Rightarrow a_{,i}^2 < a_{,0}^2 \quad \text{and} \quad \eta(\nabla a, \nabla T) = -a_{,0} < 0, \tag{5.28}$$

where T is a temporal function and is typically chosen as $T = x^0$, such that ∇T represents a time-like vector. In the above equations, we have employed a flat Lorentzian metric $\eta_{\mu\nu} = \text{diag}(-, +, +, +, ...(n-1)$ terms). If the conditions (5.28) are violated at any point on the manifold, it follows from the continuity of the derivative that they will also be false in some neighborhood of that point. By utilizing $\mathcal{J} = i\gamma^0$ and $\mathcal{D} = -i\gamma^\mu \partial_\mu$, as mentioned earlier, we readily obtain,

$$\mathcal{J}[\mathcal{D}, a] = -a_{,0} + K; \quad K := \gamma^0 \gamma^i a_{,i} \tag{5.29}$$

Now, it can be shown that K satisfies $K^2 = \eta^{ij} \partial_i a \partial_j a$, implying that the spectrum of K is essentially given by Spec$K = \pm \|\eta^{ij} \partial_i a \partial_j a\|^{\frac{1}{2}}$. As the reduced metric η^{ij} is positive definite everywhere (in our case, it is simply the identity matrix), the matrix $\mathcal{J}[\mathcal{D}, a]$ is point-wise negative. This readily verifies (5.27).

An important subspace of causal functions is constituted by the so-called *steep functions*, which satisfy the property,

$$\eta(\nabla a, \nabla a) < -1. \tag{5.30}$$

A corresponding algebraic version of causality of this steep functions can be shown to be given by[2]

$$\forall \phi \in \mathcal{H}, \langle \phi, \mathcal{J}([\mathcal{D}, a] \pm i\chi)\phi \rangle < 0 \tag{5.31}$$

An elementary proof of this algebraic version has been presented in [151,154] for an even-dimensional flat Lorentzian manifold. The key concept employed in the proof is to consider the product manifold $\tilde{\mathcal{M}} = \mathcal{M} \times \mathbb{R}$ equipped with the corresponding metric

$$\tilde{\eta} = \left(\begin{array}{c|c} \eta & 0 \\ \hline 0 & 1 \end{array} \right) \tag{5.32}$$

and introduce the function $\tilde{a} := a - x^n \in C^1(\tilde{\mathcal{M}}, \mathbb{R})$. The condition for $\tilde{a}$ to be causal in $(\tilde{\mathcal{M}}, \tilde{\eta})$ then implies that a is a steep function in $(\mathcal{M}, \mathbb{R})$.
The above result can be generalized to non-commutative (NC) spaces. To achieve this, we introduce an appropriate NC algebra $\mathcal{A}$ and replace the real function a with suitable Hermitian elements $\hat{a} \in \tilde{\mathcal{A}}$, where $\tilde{\mathcal{A}}$ is the unitized version of $\mathcal{A}$, fulfilling,

$$\forall \phi \in \mathcal{H}, \langle \phi, \mathcal{J}([\mathcal{D}, \pi(\hat{a})] \pm i\chi)\phi \rangle < 0 \tag{5.33}$$

With the algebraic condition (5.33) that captures the space of steep functions on the manifold, we can now utilize it as a "ball condition" in the Lorentzian manifold. To further define our framework, we introduce a convex cone $\mathcal{C}$ as the set of causal algebra elements that satisfy (5.27) along with an

[2] Note that the equality signs in the inequalities (5.27,5.31) have been omitted here.This is in contrast to [152,154,155]. The reason for this is explained in the sequel (see section-5.3.1).

additional condition,

$$\overline{Span_{\mathbb{C}}\mathcal{C}} = \overline{\widetilde{\mathcal{A}}} \tag{5.34}$$

By this a partial order relation is induced on the space of states $S(\mathcal{A})$, which is called the causal relation [156],

$$\forall\ \omega_1, \omega_2\ \in\ S(\mathcal{A}), \qquad \omega_1 \preceq \omega_2 \qquad \text{iff}\ \forall\ a\ \in\ \mathcal{C},\ \ \omega_1(a) \leq \omega_2(a) \tag{5.35}$$

5.3 Construction of Dirac operator, Krein space and Ball condition

In this section, we aim to replicate our Hilbert-Schmidt operatorial formulation described in section-5.1 to calculate distances in commutative and NC Lorentzian planes. Both spaces share the same structure of the Dirac operator $\mathcal{D} = \gamma^\mu \otimes \hat{P}_\mu$, which naturally acts on the Hilbert spaces $\mathbb{C}^2 \otimes L^2(\mathbb{R}^{1,1})$ and $\mathbb{C}^2 \otimes \mathcal{H}_q$ for the commutative and NC planes, respectively. A generic spinor in this Hilbert space can be represented as $\Psi = \begin{pmatrix} |\psi_1) \\ |\psi_2) \end{pmatrix}$, with $|\psi) \in L^2(\mathbb{R}^{1,1})$ and $\mathcal{H}_q$ in the respective cases.

For a pseudo-Euclidean signature (-,+) of our space-time $\mathbb{R}^{1,1}$, we can choose $\gamma^0 = -i\sigma^1$ and $\gamma^1 = \sigma^2$, where $(\gamma^0)^2 = -1$, $(\gamma^1)^2 = 1$, and $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. With these choices, the Dirac operator takes the following form:

$$\mathcal{D} = -i \begin{pmatrix} 0 & \hat{P}_0 + \hat{P}_1 \\ \hat{P}_0 - \hat{P}_1 & 0 \end{pmatrix} \tag{5.36}$$

The above form of the Dirac operator is not self-adjoint. However, we can transform the Hilbert space into a Krein space, where the operator will become a skew Krein self-adjoint operator. To proceed, let us first recall the definition of a Krein space and examine a simple example of $\mathbb{C}^2$. By constructing the Krein counterpart of $\mathbb{C}^2$, we will subsequently extend it to construct the Krein counterpart of the entire Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathcal{H}_q$ which featured in the aforementioned spectral triple.

Definition of Krein space:

If for an indefinite inner product space $\mathcal{K}$, we can consider a decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$, where $\mathcal{K}_+$ and $\mathcal{K}_-$ are mutually orthogonal, complete (with respect to the induced norm), and possess positive and negative definite inner products, respectively and if further the inner product on $\mathcal{K}$ is non-degenerate, such a space $\mathcal{K}$ is referred to as a Krein space [152, 154].

For every such decomposition, we can associate a fundamental symmetry operator $\mathcal{J}$ defined as $\mathcal{J} = \text{id}_+ \oplus (-\text{id}_-)$, satisfying the property $\mathcal{J}^2 = 1$. The operator $\mathcal{J}$ enables the transformation of the subspace with a negative inner product $(\cdot, \cdot)$ into a subspace with a positive definite inner product $\langle \cdot, \cdot \rangle$, and vice versa. This transformation is achieved by inserting the operator $\mathcal{J}$ as, $(\cdot, \mathcal{J}\cdot) = \langle \cdot, \cdot \rangle$, or equivalently, $\langle \cdot, \mathcal{J}\cdot \rangle = (\cdot, \cdot)$.

Converting $\mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathcal{H}_q$ into Krein spaces

The 2 dimensional complex vector space $\mathbb{C}^2$ can be regarded as direct sum of two sub-spaces $\mathbb{C}^1_\pm$

as follows:

$$\mathbb{C}^2 = \text{Span}\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right\} = \text{Span}\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right\} \oplus \text{Span}\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right\} := \mathbb{C}^1_+ \oplus \mathbb{C}^1_- \quad (5.37)$$

with associated projectors $\mathbb{P}^\pm = \frac{1}{2}(\mathbf{1} \pm \sigma_1)$ projecting a generic two component vector $\begin{pmatrix}\alpha\\\beta\end{pmatrix} \in \mathbb{C}^2$ to the subspaces $\mathbb{C}^1_+$and $\mathbb{C}^1_-$ respectively where both the subspaces are naturally endowed with positive definite inner products. Let us consider the fundamental symmetry operator $\mathcal{J} = \sigma_1$, where σ_1 is one of the Pauli matrices in their standard representation. This choice of $\mathcal{J}$ is supposed to render the subspace $\mathbb{C}^1_-$ as a negative definite inner product space, while preserving the positive-definiteness of the $\mathbb{C}^1_+$ subspace. It can be easily seen to be achieved by employing the following insertion :

$$\langle \cdot, \mathcal{J} \cdot \rangle = \langle \cdot, \sigma_1 \cdot \rangle := (\cdot, \cdot). \quad (5.38)$$

So now with the new inner product $(\cdot, \cdot)$ defined in (5.38) we can call our $\mathbb{C}^2$ space as a Krein space.

With this it can now be shown trivially that $\mathbb{C}^2 \otimes \mathcal{H}_q$ equipped with the inner product $\langle \cdot, \mathcal{J} \cdot \rangle$, where the used fundamental symmetry operator now is

$$\mathcal{J} = \sigma_1 \otimes \mathbf{I}, \quad (5.39)$$

is an indefinite inner product space or Krein space where $\mathbb{C}^2 \otimes \mathcal{H}_q$ splits as

$$\mathbb{C}^2 \otimes \mathcal{H}_q = (\mathbb{C}^1_+ \otimes \mathcal{H}_q) \oplus (\mathbb{C}^1_- \otimes \mathcal{H}_q).$$

With this fundamental symmetry $\mathcal{J}$, it can indeed be checked quite easily that $\mathcal{D}^* = -\mathcal{J}\mathcal{D}\mathcal{J}$ proving $\mathcal{D}$ (5.36) to be skew Krein self-adjoint.

The grading operator, as defined in section-5.2.1, simplifies to $\chi = \sigma^3 \otimes \mathbf{1}$. For a suitable representation of the algebra element a, we choose $\pi(a) = \text{diag}(a, a)$. So far, all the elements of the spectral triples are considered to be generic and applicable to both commutative and non-commutative Lorentzian planes. Based on (5.24), a general ball condition can be derived as follows,

$$\mathbf{B} = \langle \Psi, \mathcal{J}\{[\mathcal{D}, \pi(a)] + i\chi\}\Psi\rangle = \begin{pmatrix}\langle\psi_1| & \langle\psi_2|\end{pmatrix} \begin{pmatrix}-i[\hat{P}_0 - \hat{P}_1, a] & -i \\ i & -i[\hat{P}_0 + \hat{P}_1, a]\end{pmatrix} \begin{pmatrix}|\psi_1\rangle \\ |\psi_2\rangle\end{pmatrix}$$
$$< 0 \ \forall \ \Psi \in \mathcal{H} \quad (5.40)$$

It can be re-cast as,

$$\mathbf{B} = -i\langle\psi_1|[\hat{P}_0 - \hat{P}_1, a]|\psi_1\rangle - i\langle\psi_2|[\hat{P}_0 + \hat{P}_1, a]|\psi_2\rangle + i[\langle\psi_2|\psi_1\rangle - \langle\psi_1|\psi_2\rangle] < 0 \quad (5.41)$$

Above equation can be taken as a master equation or principal ball condition. We now specialize into two separate cases for commutative and non-commutative planes.

5.3.1 Distance in commutative Lorentzian plane

As a warm up exercise we first undertake the computation of the distance on the (1+1) dimensional flat commutative Lorentzian manifold. The spectral triple for commutative Lorentzian manifold $\mathbb{R}^{1,1}$ is defined through the algebra of smooth functions over $\mathbb{R}^{1,1}$: $\mathcal{A} = C_0^\infty(\mathbb{R}^{1,1})$, a Hilbert space [3] $\mathcal{H} = \mathbb{C}^2 \otimes L^2(\mathbb{R}^{1,1})$, consisting of generic spinor like $\Psi = \begin{pmatrix} |\psi_1) \\ |\psi_2) \end{pmatrix}$ where $|\psi_i)$ are abstract kets whose representation in $|t,x\rangle$ basis becomes $L^2(\mathbb{R}^{1,1})$ element. With the Dirac operator already defined in previous section we arrived at a simplified ball condition (5.41). Inserting completeness relation of $|t,x\rangle \equiv |x\rangle$ basis: $\int dt\, dx\, |t,x\rangle\langle t,x| = \mathbf{1}$, (5.41) can be rewritten as,

$$-i \int d^2x\, d^2y\, \psi_1^*(x)\psi_1(y)\langle x|[\hat{P}_0 - \hat{P}_1, a]|y\rangle - i \int d^2x\, d^2y\, \psi_2^*(x)\psi_2(y)\langle x|[\hat{P}_0 + \hat{P}_1, a]|y\rangle$$
$$+ i \int d^2x[\psi_2^*(x)\psi_1(x) - \psi_1^*(x)\psi_2(x)] < 0 \tag{5.42}$$

Here the action of momentum operator on the C^∞ function $a(t,x) := a(x)$ takes place through commutator bracket and it gives,

$$\langle x|[\hat{P}_\mu, a(x)]|y\rangle = -i\partial_\mu a(x)\delta^{(2)}(x-y) \tag{5.43}$$

With this, (5.42) further simplies as

$$\int d^2x\ [(\partial_0 a)(-|\psi_1(x)|^2 - |\psi_2(x)|^2) + (\partial_1 a)(|\psi_1(x)|^2 - |\psi_2(x)|^2)]$$
$$< i \int d^2x\, [\psi_1^*(x)\psi_2(x) - \psi_2^*(x)\psi_1(x)] \tag{5.44}$$

Since the values of $\psi_1(x)$ and $\psi_2(x)$ are fully arbitrary, we may pick them to be compactly supported in an arbitrarily small neighborhood around any given point and conclude that this inequality should hold for the integrands themselves and eventually should remain valid point-wise. (see Theorem11 in [154]). As a result, we can write.,

$$(\nabla_\mu a)V^\mu < i(\psi_1^*\psi_2 - \psi_2^*\psi_1)\ ; V_0 = -V^0 = |\psi_1|^2 + |\psi_2|^2, V_1 = V^1 = |\psi_1|^2 - |\psi_2|^2 \tag{5.45}$$

Here, $V = (V^0, V^1)$ is a timelike two-vector, and $a \in \mathcal{A}$ belongs to the set of steep functions, i.e., $\eta(\nabla a, \nabla a) < -1$. This implies that ∇a is also a timelike vector, and $(V^\mu\partial_\mu a)$ is intrinsically negative[4] This allows us to rewrite (5.45) as follows:

$$|V^\mu\partial_\mu a| = -(V^\mu\partial_\mu a) > i(\psi_1\psi_2^* - \psi_2\psi_1^*) = -2|\psi_1||\psi_2|\sin(\alpha_1 - \alpha_2) \tag{5.46}$$

where the phases α_1 and α_2 are defined as $\psi_1 = |\psi_1|e^{i\alpha_1}$ and $\psi_2 = |\psi_2|e^{i\alpha_2}$. We must now look for the most suitable algebraic element for which the infimum is achieved since the calculation of the Lorentzian distance in (5.23) entails the computation of the infimum. It will be better to do a minor

[3]Here the $L^2(\mathbb{R}^{1,1})$ part of $\mathcal{H}$ is actually analogous to $\mathcal{H}_q$ in our HS formulation.

[4]Note that if equality signs were included in (5.31), $(V^\mu\nabla_\mu a)$ would not have been intrinsically negative, and we would be forced to consider the case $V^\mu\nabla_\mu a = 0$ as well. For V^μ timelike, this would have implied that $\nabla_\mu a$ is purely spacelike, which is a scenario we want to avoid. In any case, the inclusion or exclusion of equality signs does not impact the computation of supremum or infimum, as one is a dense subspace of the other.

variation of the inequality (5.42, 5.46) as

$$\mathbf{B} \leq 0 \text{ i.e. } |V^\mu \partial_\mu a| \geq -2|\psi_1||\psi_2| \sin(\alpha_1 - \alpha_2) \tag{5.47}$$

The inclusion of the equality sign here will not have any effect on the computation of the infimum, as the set defined by (5.42,5.46) is a dense subset of (5.47), thus adding the equality sign here will not change how the infimum is calculated. The saturation condition in (5.47) is now fulfilled by first holding $|\psi_1|$ and $|\psi_2|$ fixed and varying α_1, α_2. Clearly the maximal value reached by the R.H.S above will correspond to $2|\psi_1||\psi_2|$ with the choice $\alpha_2 - \alpha_1 = \frac{\pi}{2}$ and (5.47) reduces to the following form

$$|V^\mu \partial_\mu a| \geq 2|\psi_1||\psi_2| \tag{5.48}$$

Correspondingly the generic form of the spinor ψ which maximizes the R.H.S is $\begin{pmatrix} |\psi_1| \\ i|\psi_2| \end{pmatrix} e^{i\alpha_1}$. On the other hand we can now apply *reverse* Cauchy-Schwarz inequality [151] to write

$$|V^\mu \partial_\mu a| \geq \|\nabla_\mu a\|_L \|V^\mu\|_L. \tag{5.49}$$

Note that here L in the subscript is a reminder of Lorentzian norm defined as $\|v\|_L = \sqrt{-\eta(v,v)}$ for a time-like vector v. In the next stage , we vary $|\psi_1|$ and $|\psi_2|$, so that V becomes collinear with ∇a and (5.49) becomes,

$$|V^\mu \partial_\mu a| = \|\nabla_\mu a\|_L \|V^\mu\|_L \tag{5.50}$$

We now make use of (5.45), so that by (5.47) and (5.50) we finally get

$$\|\nabla a\|_L \geq 1 \tag{5.51}$$

We therefore look for a solution of a satisfying

$$\|\nabla a\|_L = 1 \tag{5.52}$$

Equivalently , the search can be restricted to the solution set of the following differential equation satisfied by real functions $a(x,t)$

$$\left(\frac{\partial a}{\partial t}\right)^2 - \left(\frac{\partial a}{\partial x}\right)^2 = 1 \tag{5.53}$$

yielding the following one-parameter (λ) solution set of $a(t,x)$:

$$a(t,x) = t\cosh\lambda + x\sinh\lambda \tag{5.54}$$

By (5.23) the distance between two points say $P(t_1,x_1), Q(t_2,x_2)$ in commutative Lorentzian manifold (where P preceeds Q chronologically i.e. P $\prec$ Q) is now given by,

$$d(\delta_P, \delta_Q) = \inf_{a\in\mathcal{B}}\{[\delta_Q(a) - \delta_P(a)]^+\} = \inf_{a\in\mathcal{B}}\{[a(t_2,x_2) - a(t_1,x_1)]^+\} \tag{5.55}$$

where δ_P, δ_Q are pure states representing the respective events (t_1,x_1) and (t_2,x_2) in the forward light-cone of the commutative plane. Eq (5.55) can now be recast as,

$$d(\delta_P, \delta_Q) = \inf_\lambda\{[(t_2 - t_1)\cosh\lambda + (x_2 - x_1)\sinh\lambda]^+\} \tag{5.56}$$

which on minimisation with respect to λ (to get the infimum value of the set) gives ,

$$d(\delta_P, \delta_Q) = \sqrt{(t_2 - t_1)^2 - (x_2 - x_1)^2} \tag{5.57}$$

Note that the causal information has been incorporated in the resulting distance in (5.56). The quantity in the parentheses of (5.56) will undoubtedly be negative if the events are not related causally, that is, if P does not occur prior Q [5] and the distance equation will give a value of zero, showing that the distance is not symmetric when a pair of events are interchanged.

Functional Derivative Approach

In this part, we will address the same problem using a functional differentiation approach, which we will find to be easier to execute and adaptable to the computation of distances in the Lorentzian Moyal plane, to be discussed in the next sub-section. To begin with, let us consider the left-hand side of the ball condition (5.41), denoted as $\mathbf{B}[\psi_1, \psi_1^*, \psi_2, \psi_2^*]$, which can now be treated as a functional of ψ_1, ψ_1^*, ψ_2, and ψ_2^*:

$$\begin{aligned}\mathbf{B}[\psi_1, \psi_1^*, \psi_2, \psi_2^*] = -\int d^2x\, \big[|\psi_1(x)|^2(\partial_0 a - \partial_1 a) + |\psi_2(x)|^2(\partial_0 a + \partial_1 a) \\ + i\{\psi_1^*(x)\psi_2(x) - \psi_2^*(x)\psi_1(x)\}\big]\end{aligned} \tag{5.58}$$

Clearly the algebra element $a \in \mathcal{A}$ for which the infimum in (5.23) is reached giving the distance should be such that

$$\sup_{\psi_1, \psi_2, \psi_1^*, \psi_2^*} \mathbf{B}[\psi_1, \psi_2, \psi_1^*, \psi_2^*] = 0 \tag{5.59}$$

To put it another way, we wish to identify the algebra element for which the functional $\mathbf{B}[\psi_1, \psi_1^*, \psi_2, \psi_2^*]$ gets its supremum value of zero. In order to demonstrate that the maximum of $\mathbf{B}[\psi_1, \psi_1^*, \psi_2, \psi_2^*]$ corresponds to the zero value that coincides with the supremum, a suitable constraint must first be imposed on the algebra element $a \in \mathcal{A}$, which arises as an offshoot. The following matrix equation results from functionally differentiating the functional $\mathbf{B}[\psi_1, \psi_1^*, \psi_2, \psi_2^*]$ with respect to $\psi_1(x), \psi_2(x)$,

$$\begin{pmatrix} -(\partial_0 - \partial_1)a(x) & i \\ -i & -(\partial_0 + \partial_1)a(x) \end{pmatrix} \begin{pmatrix} \psi_1^*(x) \\ \psi_2^*(x) \end{pmatrix} = 0 \tag{5.60}$$

The analogous matrix equation, which is merely the complex conjugate of the previous equation, is produced by the variation with respect to the $\psi_1^*(x)$ and $\psi_2^*(x)$ variables. The equations are fulfilled quite trivially, if ψ_1^*, ψ_2^* are zero. So for non-vanishing values of ψ_1^*, ψ_2^* to exist, satisfying the above equations, it requires a vanishing determinant of the coefficient matrix in (5.60), yielding a condition on $a(x)$, which is precisely (5.53), giving (5.54) for $a(t, x)$.
Let us now rewrite $\mathbf{B}$, using (5.54) as,

$$\mathbf{B} = -\int d^2x\, [e^{-\lambda}\psi_1^*(x)\psi_1(x) + e^{\lambda}\psi_2^*(x)\psi_2(x) + i\{\psi_1^*(x)\psi_2(x) - \psi_2^*(x)\psi_1(x)\}] \tag{5.61}$$

[5] For $P \prec Q$ i.e. $t_2 > t_1$ and $(t_2 - t_1)^2 > (x_2 - x_1)^2$. Additionally, $|\cosh\lambda| > |\sinh\lambda|$, $\forall\lambda$. Whole quantity in (5.56) will thus always be positive.

The following scaling transformations, $\psi_1(x) \to \psi_1'(x) := e^{-\frac{\lambda}{2}}\psi_1(x)$; $\psi_2(x) \to \psi_2'(x) := e^{\frac{\lambda}{2}}\psi_2(x)$ allows us to write **B** in a simplified form as

$$\mathbf{B} = -\int d^2x\,[\psi_1'^*(x)\psi_1'(x) + \psi_2'^*(x)\psi_2'(x) + i\{\psi_1'^*(x)\psi_2'(x) - \psi_2'^*(x)\psi_1'(x)\}] \tag{5.62}$$

We now carry out a linear transformation of the following form which enables us to write **B** in terms of independent functions $\phi_\pm(x)$, defined as,

$$\begin{pmatrix}\phi_+(x)\\ \phi_-(x)\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & i\\ -1 & i\end{pmatrix}\begin{pmatrix}\psi_1'(x)\\ \psi_2'(x)\end{pmatrix} \tag{5.63}$$

A simple algebra then shows that **B** is really independent of $\phi_+(x)$ and depends on $\phi_-(x)$ only as,

$$\mathbf{B} = -2\int d^2x|\phi_-(x)|^2 \tag{5.64}$$

Equation (5.64) clearly suggests that **B** reaches its supremum value zero for $|\phi_-| = 0$ since it depicts an inverted parabola in the $(\mathbf{B}, |\phi_-|)$ plane. Therefore, regardless of whether ϕ_+ or ϕ_- are present, the form of the algebra element (5.54) automatically assures that **B** cannot exceed zero. We finally reach the same distance function (5.57) as a result.

5.3.2 Distance in 2D Lorentzian Moyal plane

In this penultimate part, we finally begin the computation of the distance in the NC scenario. Note that the same Hilbert space $\mathcal{H}_c$ constructed in (3.28) will again provide a representation of the NC 2-dimensional Moyal plane algebra: $[\hat{t}, \hat{x}] = i\theta$ with Lorentzian signature (-,+). Here, the *vacuum* $|0\rangle$ and the creation and annihilation operators are described analogously as:

$$\hat{b} = \frac{\hat{t} + i\hat{x}}{\sqrt{2\theta}},\ \hat{b}^\dagger = \frac{\hat{t} - i\hat{x}}{\sqrt{2\theta}};\ \ \hat{b}|0\rangle = 0 \tag{5.65}$$

However there are certain differences as well. To see this note that although the vacuum transforms under space-time translation as,

$$|0\rangle \to |z\rangle = U(z,\bar{z})|0\rangle;\ \ U(z,\bar{z}) = e^{-\bar{z}\hat{b} + z\hat{b}^\dagger} \tag{5.66}$$

along with all the raising/lowering operators transforming adjointly as :

$$\begin{aligned}&\hat{b} \to \hat{b}_z = \hat{b} - z = U(z,\bar{z})\,\hat{b}\,U^\dagger(z,\bar{z});\ \ \hat{b}^\dagger \to \hat{b}_z^\dagger = \hat{b}^\dagger - \bar{z} = U(z,\bar{z})\,\hat{b}^\dagger\,U^\dagger(z,\bar{z});\\ &\rho_0 \to \rho_z = |z\rangle\langle z| = U\rho_0\,U^\dagger;\ \mathcal{D} \to \mathcal{D}_z = U(z,\bar{z})\,\mathcal{D}\,U^\dagger(z,\bar{z})\end{aligned} \tag{5.67}$$

with $U(z,\bar{z})$ (5.66) being the same as its Euclidean counter-part, the situation is different when the Lorentz boost is concerned. In fact the transformation under Lorentz boost,

$$\begin{pmatrix}\hat{t}\\ \hat{x}\end{pmatrix} \to \begin{pmatrix}\hat{t}'\\ \hat{x}'\end{pmatrix} = \begin{pmatrix}\cosh\phi & \sinh\phi\\ \sinh\phi & \cosh\phi\end{pmatrix}\begin{pmatrix}\hat{t}\\ \hat{x}\end{pmatrix}, \tag{5.68}$$

is unitarily implemented on Hilbert space $\mathcal{H}_c$ through adjoint action as:

$$\hat{b} \to \hat{b_\phi} = \cosh\phi\,\hat{b} + i\sinh\phi\,\hat{b}^\dagger = U(\phi)\hat{b}U^\dagger(\phi); \quad \hat{b}^\dagger \to \hat{b_\phi}^\dagger = \cosh\phi\,\hat{b}^\dagger - i\sinh\phi\,\hat{b} = U(\phi)\hat{b}^\dagger U^\dagger(\phi)$$
$$\rho_0 \to \rho_\phi = |\phi\rangle\langle\phi| = U(\phi)\rho_0 U^\dagger(\phi); \;\; \mathcal{D} \to \mathcal{D}_\phi = U(\phi)\mathcal{D}U^\dagger(\phi) \tag{5.69}$$

Here, we introduce the squeezing operator $U(\phi) = e^{-\frac{i\phi}{2}(\hat{b}^2 + \hat{b^\dagger}^2)}$, which is responsible for the transformation. It is worth noting that the coherent state $|z\rangle$ also undergoes a transformation as $|z\rangle \to |z;\phi\rangle = U(\phi)|z\rangle$. This holds true for all vectors $|\psi\rangle \in \mathcal{H}_c$ that satisfy $b_\phi|z;\phi\rangle = z|z;\phi\rangle$. In particular, the *vacuum* state $|0\rangle \in \mathcal{H}_c$, which is also an element of the coherent state family with $z = 0$, is not invariant under this boost: $|0;\phi\rangle = U(\phi)|0\rangle \neq |0\rangle$. Similarly, its associated pure state $|0;\phi\rangle\langle 0;\phi| \neq |0\rangle\langle 0|$. This is in contrast to the Euclidean case where the vacuum $|0\rangle$ simply acquires a phase under SO(2) rotation, resulting in the associated pure state $|0\rangle\langle 0|$ remaining invariant. It is important to note that despite this discrepancy, the space-time algebra $[\hat{t}, \hat{x}] = i\theta$, along with the ball condition, remains invariant under this Lorentz transformation. Nonetheless, we can easily demonstrate the following identity:

$$\langle z,\phi|\hat{a}_{w,\phi}|z,\phi\rangle - \langle w,\phi|\hat{a}_{w,\phi}|w,\phi\rangle = \langle z-w|\hat{a}|z-w\rangle - \langle 0|\hat{a}|0\rangle; \;\; \hat{a}_{z,\phi} := U(\phi)\hat{a}_z U^\dagger(\phi) \tag{5.70}$$

so that the spectral distance is invariant under both translation and Lorentz boost:

$$d_{\mathcal{D}}(\rho_0, \rho_{z-w}) = d_{\mathcal{D}_{z,\phi}}(\rho_{w,\phi}, \rho_{z,\phi}) \tag{5.71}$$

demonstrating how distance is invariant under the Poincare transformation. Three generators, such as two translations and one boost for the transformation, are present in (1+1) dimension and together they create a closed $\mathfrak{iso}(1,1)$ algebra.

Finally, we start the explicit distance computation. We will essentially use the same HS formalism for this as was done for the Euclidean Moyal plane in chapter-2. We construct the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ here with the algebra $\mathcal{A} = \mathcal{H}_q$, the Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathcal{H}_q$, and the Dirac operator $\mathcal{D} = \gamma^\mu \otimes \hat{P}_\mu$ (5.36) which again by default acts on $\mathbb{C}^2 \otimes \mathcal{H}_q$. Here, an element $|\psi) \in \mathcal{H}_q$ is adjointly acted upon by the momentum operators as follows:

$$\hat{P}_i|\psi) = \frac{\epsilon_{ij}}{\theta}|[\hat{x}_j, \psi]); \;\; i,j \in [0,1] \tag{5.72}$$

The commutators occurring in the ball condition (5.41) now becomes,

$$[\hat{P}_0 + \epsilon\hat{P}_1, \hat{a}] = \frac{1}{\theta}[\hat{x} - \epsilon\hat{t}, \hat{a}]; \;\; \epsilon = \pm 1, \;\; a \in \mathcal{A} = \mathcal{H}_q \tag{5.73}$$

It is now clear that $\hat{x}, \hat{t}$ has normal left action on $\mathcal{H}_c$, so that the matrix elements corresponding to the first two terms in (5.41) are now well defined with $|\psi_1\rangle$ and $|\psi_2\rangle \in \mathcal{H}_c$. In contrast to the commutative situation, we don't have any notion of a point or a clearly defined event. So we use coherent states, such as (2.12), which have the advantageous property of being maximally localised space-time events: $\Delta\hat{t}\Delta\hat{x} = \frac{\theta}{2}$. Therefore, it will be desirable to use the Bargmann-Fock formulation of coherent state basis (Appendix-F) once more. By inserting completeness relation (F.2) in the ball condition (5.41) this gives the functional,

$$\mathbf{B}[\psi_1,\psi_1^*,\psi_2,\psi_2^*] = -i\int d\mu(z,\bar{z})\,d\mu(w,\bar{w})\,[\psi_1^*(z)\psi_1(\bar{w})\langle\bar{z}|[\hat{P}_0-\hat{P}_1,a]|w\rangle + \psi_2^*(z)\psi_2(\bar{w})\langle\bar{z}|[\hat{P}_0+\hat{P}_1,a]|w\rangle] + i\int d\mu(z,\bar{z})\,[\psi_2^*(z)\psi_1(\bar{z})-\psi_1^*(z)\psi_2(\bar{z})] < 0 \tag{5.74}$$

Using (5.73) we have,

$$\mathbf{B}[\psi_1,\psi_1^*,\psi_2,\psi_2^*] = -\frac{i}{\theta}\int d\mu(z,\bar{z})\,d\mu(w,\bar{w})\,[\psi_1^*(z)\psi_1(\bar{w})\langle\bar{z}|[\hat{t}+\hat{x},\hat{a}]|w\rangle - \psi_2^*(z)\psi_2(\bar{w})\langle\bar{z}|[\hat{t}-\hat{x},\hat{a}]|w\rangle] + i\int d\mu(z,\bar{z})\,[\psi_2^*(z)\psi_1(\bar{z})-\psi_1^*(z)\psi_2(\bar{z})] < 0 \tag{5.75}$$

We take the left hand side of (5.75) as a functional $\mathbf{B}[\psi_1,\psi_1^*,\psi_2,\psi_2^*]$ of $\psi_1(\bar{z}),\psi_1^*(z),\psi_2(\bar{z}),\psi_2^*(z)$. In order to calculate the optimal algebra element which saturates the ball condition we maximise **B** with respect to $\psi_1(\bar{u})$ and $\psi_2(\bar{u})$ using (F.6) and equate them to zero, to arrive respectively at the following pair of equations:

$$\begin{aligned}\frac{1}{\theta}\langle\psi_1|[\hat{t}+\hat{x},\hat{a}]|u\rangle &= \langle\psi_2|u\rangle\\ \frac{1}{\theta}\langle\psi_2|[\hat{t}-\hat{x},\hat{a}]|u\rangle &= \langle\psi_1|u\rangle\end{aligned} \tag{5.76}$$

As can be checked easily that the differentiation w.r.t. $\psi_1^*(u)$ and $\psi_2^*(u)$ just yield an equivalent pair of equations related to (5.76) by just a complex conjugation. Combining this pair of equations by using the completeness relation (F.2), one readily obtains a condition on the algebra element as,

$$\frac{1}{\theta^2}[\hat{t}+\hat{x},\hat{a}]\,[\hat{t}-\hat{x},\hat{a}] = 1 \tag{5.77}$$

The general form of the optimal algebra element $\hat{a}_s$, which is an infinite dimensional matrix satisfying (5.77) and can also ensure ISO(1,1) invariance of the resulting distance, is: $\hat{a} = \alpha\hat{t}+\beta\hat{x}$ (with $\alpha,\beta\in\mathbb{R}$ for $\hat{a}$ to be hermitian). The general form of the algebra element is ultimately reached by substituting this in (5.77):

$$\hat{a}_\lambda = \hat{t}\cosh\lambda + \hat{x}\sinh\lambda;\quad \lambda\in\mathbb{R} \tag{5.78}$$

To see it more concretely, let us write (5.77) in the form of differential equations.For that we consider the matrix elements of the above operator equation (5.77) in coherent state basis as,

$$\int d\mu(w,\bar{w})\langle\bar{z}|[e^{\alpha}\hat{b}+e^{-\alpha}\hat{b}^\dagger,\hat{a}]|w\rangle\langle\bar{w}|[e^{-\alpha}\hat{b}+e^{\alpha}\hat{b}^\dagger,\hat{a}]|u\rangle = \theta\int d\mu(w,\bar{w})\langle\bar{z}|w\rangle\langle\bar{w}|u\rangle \tag{5.79}$$

where $\alpha = -\frac{i\pi}{4}$. Also note that we have inserted the resolution of identity (F.2) here. Now using

$$\begin{aligned}\langle\bar{z}|[\hat{b},\hat{a}]|w\rangle &= \partial_{\bar{z}}a(\bar{z},w) - wa(\bar{z},w)\\ \langle\bar{z}|[\hat{b}^\dagger,\hat{a}]|w\rangle &= \bar{z}a(\bar{w},z) - \partial_w a(\bar{z},w)\end{aligned}$$

and writing $\langle\bar{w}|\hat{a}|z\rangle = a(\bar{w},z) = f(\bar{w},z)e^{\bar{w}z}$ (for normal ordered operator $\hat{a}$, see (2.10)), we can further simplify the above equation as,

$$\int d\mu(\bar{w},w)e^{\bar{z}w+\bar{w}z}\left(e^{\alpha}\partial_{\bar{z}}f(\bar{z},w)-e^{-\alpha}\partial_{w}f(\bar{z},w)\right)\left(e^{-\alpha}\partial_{\bar{w}}f(\bar{w},u)-e^{\alpha}\partial_{u}f(\bar{w},u)\right)$$
$$=\theta\int d\mu(w,\bar{w})e^{\bar{z}w+\bar{w}z} \tag{5.80}$$

By comparing the coefficients of $e^{\bar{z}w+\bar{w}z}$ in the integrands of either sides of this equation, we can easily see that the following set of differential equations are necessarily satisfied by the function $f(\bar{z},w)$:

$$\begin{aligned} e^{\alpha}\partial_{\bar{z}}f(\bar{z},w)-e^{-\alpha}\partial_{w}f(\bar{z},w)&=\sqrt{\theta}\;pe^{\lambda}\\ e^{-\alpha}\partial_{\bar{w}}f(\bar{w},u)-e^{\alpha}\partial_{u}f(\bar{w},u)&=\sqrt{\theta}\;p^{*}e^{-\lambda}\end{aligned} \tag{5.81}$$

where $p\in\mathbb{C}$ with $|p|=1$ and $\lambda\in\mathbb{R}$. This ensures that the product of these two expressions, occurring in the left hand side of this pair of equations is indeed θ. We now replace the variable $\bar{w}$ by $\bar{z}$ and u by w in the second equation of (5.81) and write the equations in matrix form as,

$$\begin{pmatrix} e^{\alpha} & -e^{-\alpha}\\ e^{-\alpha} & -e^{\alpha}\end{pmatrix}\begin{pmatrix}\partial_{\bar{z}}f\\ \partial_{w}f\end{pmatrix}=\sqrt{\theta}\begin{pmatrix}pe^{\lambda}\\ p^{*}e^{-\lambda}\end{pmatrix} \tag{5.82}$$

Inverting the above coefficient matrix and solving for $\partial_{\bar{z}}f$ and $\partial_{w}f$ we get,

$$\begin{aligned}\partial_{\bar{z}}f&=\frac{\sqrt{\theta}}{2}e^{\frac{i\pi}{4}}\left(pe^{\lambda}-ip^{*}e^{-\lambda}\right)=K_1\\ \partial_{w}f&=\frac{\sqrt{\theta}}{2}e^{\frac{i\pi}{4}}\left(ipe^{\lambda}-p^{*}e^{-\lambda}\right)=K_2\end{aligned} \tag{5.83}$$

So from the above equations we can infer that $f(\bar{z},w)=K_1\bar{z}+K_2w$, which in turn gives the form of $\hat{a}$ as, $\hat{a}=K_2\hat{b}+K_1\hat{b}^{\dagger}$. Now imposing the hermiticity condition on the algebra element $\hat{a}$, we simply arrive on the conclusion that $K_1=\bar{K}_2$.Equating the real and imaginary part of this equation we find $p=i$. So finally making use of this in (5.83) , we get a one parameter family of $\hat{a}$ dependent on $\lambda\in\mathbb{R}$ to be given by,

$$\hat{a}_{\lambda}=\sqrt{\frac{\theta}{2}}[\hat{b}(\cosh\lambda-i\sinh\lambda)+\hat{b}^{\dagger}(\cosh\lambda+i\sinh\lambda)]=\hat{t}\cosh\lambda+\hat{x}\sinh\lambda \tag{5.84}$$

which is (5.78) itself. We now emulate the commutative case and substitute this algebra element (5.78) in (5.72) to get,

$$\begin{aligned}\mathbf{B}[\psi_1,\psi_1^{*},\psi_2,\psi_2^{*}]=&-e^{-\lambda}\int d\mu(z,\bar{z})\psi_1^{*}(z)\psi_1(\bar{z})-e^{\lambda}\int d\mu(z,\bar{z})\psi_2^{*}(z)\psi_2(\bar{z})\\ &+i\int d\mu(z,\bar{z})[\psi_2^{*}(z)\psi_1(\bar{z})-\psi_1^{*}(z)\psi_2(\bar{z})]\end{aligned} \tag{5.85}$$

Using scaling transformation $\psi_1(\bar{z})\to\psi_1'(\bar{z}):=e^{-\frac{\lambda}{2}}\psi_1(\bar{z})$; $\psi_2(\bar{z})\to\psi_2'(\bar{z}):=e^{\frac{\lambda}{2}}\psi_2(\bar{z})$, as before we rewrite (5.85) as,

$$\mathbf{B}=-\int d\mu(z,\bar{z})[\psi_1'^{*}(z)\psi_1'(\bar{z})+\psi_2'^{*}(z)\psi_2'(\bar{z})+i\{\psi_1'^{*}(z)\psi_2'(z)-\psi_2'^{*}(z)\psi_1'(z)\}] \tag{5.86}$$

It seems that the functions on which **B** depends are not all independent. So we again use a transformation similar to (5.63) as

$$\begin{pmatrix} \psi_1(\bar{z}) \\ \psi_2(\bar{z}) \end{pmatrix} \rightarrow \begin{pmatrix} \phi_+(\bar{z}) \\ \phi_-(\bar{z}) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} \begin{pmatrix} \psi_1'(\bar{z}) \\ \psi_2'(\bar{z}) \end{pmatrix} \tag{5.87}$$

to get **B** as $\mathbf{B}[\phi_-, \phi_-^*]$ in the following way,

$$\mathbf{B} = -2 \int d\mu(z, \bar{z}) |\phi_-(z)|^2 \tag{5.88}$$

The aforementioned equation clearly demonstrates that **B** is independent of ϕ_+ and ϕ_+^*, and it also demonstrates that **B** achieves its maximum value when $|\phi_-| = 0$, which is its supremum value with the $\hat{a}$ chosen above (5.78). Similar to the Euclidean example covered in section-5.1, we can find an acceptable bound to the distance function in this situation as well, with the exception that it must be a lower limit. This fact indeed serves as evidence that we have made the proper choice. In the subsection that follows, we will derive this bound and demonstrate that the selection of $\hat{a}$ (5.78) does, in fact, cause this bound to be saturated.

Calculation of the lower bound of distance

We may attempt to obtain a bound on the separation between the pure states here, just like in the Euclidean scenario. Contrary to the previous instance, however, the distance formula includes the computation of the maximum, thus in this situation we must obtain a lower bound. We may write, using the same process (5.11) as ,

$$\begin{aligned} d(\rho_0, \rho_z) &= \int_0^1 d\mu \frac{d}{d\mu} [tr(\rho_{\mu z} \hat{a})] = \int_0^1 d\mu \, tr \left(\rho_{\mu z} [\bar{z}\hat{b} - z\hat{b}^\dagger, \hat{a}] \right) \\ &= \frac{1}{\sqrt{2\theta}} \int_0^1 d\mu \; tr \left(\rho_{\mu z} \{ [\hat{t}, \hat{a}](\bar{z} - z) + i[\hat{x}, \hat{a}](z + \bar{z}) \} \right) \end{aligned} \tag{5.89}$$

We can now parameterize t and x as

$$t = r \cosh \psi; \quad x = r \sinh \psi, \qquad \text{so that } z = \frac{r(\cosh \psi + i \sinh \psi)}{\sqrt{2\theta}} \tag{5.90}$$

representing the points on a hyperbola $t^2 - x^2 = r^2$ in the forward light cone and r is a real constant:$0 < r < \infty$. Note that with this choice of parametrization we are restricting the points in the causal cone. After some simplification, (5.89) can now be re-written as,

$$d(\rho_0, \rho_z) = \frac{ir}{\theta} \int_0^1 d\mu \, tr \left(\rho_{\mu z} [\hat{x} \cosh \psi - \hat{t} \sinh \psi, \hat{a}] \right) \tag{5.91}$$

But note that, unlike in the Euclidean case (5.12) we *cannot* just write $tr(\rho_{\mu z}[\hat{x}, \hat{a}]) \leq \|[\hat{x}, \hat{a}]\|_{op}$, as here the so-called ball condition has a completely different structure. The ball condition (5.26), just implies that the matrix $M := \mathcal{J}\{[\mathcal{D}, \pi(\hat{a})] + i\chi\}$ is a negative operator. Realizing its tensor product structure, we can re-write this, by making use of (5.72) as,

$$M = \mathcal{J}\{[\mathcal{D}, \pi(\hat{a})] + i\chi\} = -i \begin{pmatrix} \frac{1}{\theta}[\hat{t}+\hat{x},\hat{a}] & 1 \\ -1 & \frac{1}{\theta}[\hat{x}-\hat{t},\hat{a}] \end{pmatrix}$$
$$= -\mathbf{1}_2 \otimes \frac{i}{\theta}[\hat{x},\hat{a}] + \sigma_2 \otimes \mathbf{1} - \sigma_3 \otimes \frac{i}{\theta}[\hat{t},\hat{a}] \tag{5.92}$$

where $\mathbf{1}_2$ is 2×2 identity matrix. Now carrying out a suitable unitary transformation with $U = \mathbf{1}_2 \otimes U_1$, we can diagonalize the hermitian operator $-\frac{i}{\theta}[\hat{t},\hat{a}]$, such that,

$$U_1^{-1}(-\frac{i}{\theta}[\hat{t},\hat{a}])U_1 = \text{diag}(\kappa_1, \kappa_2, \cdots) := \Gamma \ ; \quad \kappa_i \in \mathbb{R} \ \forall \ i \tag{5.93}$$

so that Γ is possibly an infinite dimensional diagonal matrix and

$$M \to M' = U^{-1}MU = \mathbf{1}_2 \otimes U_1^{-1}(-\frac{i}{\theta}[\hat{x},\hat{a}])U_1 + \sigma_2 \otimes \mathbf{1} + \sigma_3 \otimes \Gamma \tag{5.94}$$

Now for each entry in the right slot of this total matrix M', say the (i,j)-th element, there is a $\mathfrak{u}(2)$ Lie algebra element and can be written as,

$$\mathbf{1}_2 \left[U_1^{-1}(-\frac{i}{\theta}[\hat{x},\hat{a}])U_1 \right]_{ij} + \sigma_2 \delta_{ij} + \sigma_3 \Gamma_{ij} \in \mathfrak{u}(2) \tag{5.95}$$

This can now be subjected to another SU(2) transformation $U = U_2 \otimes \mathbf{1}$ with $U_2 = e^{\frac{i}{2}\alpha_1\sigma_1}$, with a suitable choice of α_1, to bring it to the diagonal form

$$\mathbf{1}_2 \left[U_1^{-1}(-\frac{i}{\theta}[\hat{x},\hat{a}])U_1 \right]_{ij} + \sigma_3 \sqrt{(\delta_{ij})^2 + (\Gamma_{ij})^2} \quad ; \quad \Gamma_{ij} = \kappa_i \delta_{ij} \ \ \text{(no sum)} \tag{5.96}$$

So that after two successive transformation by $(\mathbf{1}_2 \otimes U_1)$ and $(U_2 \otimes \mathbf{1})$, the matrix M (5.92), as a whole finally transforms to

$$M \to M'' := (U_2 \otimes U_1)^{-1} M (U_2 \otimes U_1) = \mathbf{1}_2 \otimes U_1^{-1}(-\frac{i}{\theta}[\hat{x},\hat{a}])U_1 + \sigma_3 \otimes \sqrt{1 + \Gamma^\dagger \Gamma} \tag{5.97}$$

Negativity of this operator implies,

$$\mathbf{1} \otimes i\, U_1^{-1}[\hat{x},\hat{a}]U_1 - \sigma_3 \otimes \sqrt{\theta^2 + U_1^{-1}[\hat{t},\hat{a}]^\dagger [\hat{t},\hat{a}]U_1} \ > 0 \tag{5.98}$$

Now since $U_1^{-1}[\hat{t},\hat{a}]^\dagger[\hat{t},\hat{a}]U_1$ is a positive operator, we readily obtain,

$$\mathbf{1} \otimes i[\hat{x},\hat{a}] > \sigma_3 \otimes \theta \tag{5.99}$$

which by comparing two sides, further implies that

$$i[\hat{x},\hat{a}] \ > \ \theta \tag{5.100}$$

As an interesting consequence, we note that if we include the equality sign in (5.98), the saturation condition of the inequality will be fulfilled only by algebraic elements that commute with $\hat{t}$. Consequently, $\hat{a}(\hat{t},\hat{x})$ will solely depend on $\hat{t}$, leading to $[\hat{t}, \hat{a}(\hat{t},\hat{x})] = 0$. The inequality (5.100) then implies that $\hat{a}(\hat{t},\hat{x})$ must be linear in $\hat{t}$, resulting in $i[\hat{x},\hat{a}]$ being equal to $\theta\mathbf{1}$. Thus, (5.100) also determines the scale. It is noteworthy that this characteristic is absent in the Euclidean case.

At this point, it is important to highlight that this inequality, along with the aforementioned observations, must hold true in all Lorentz frames. Since the original ball condition (5.40) is a Lorentz invariant quantity, we can utilize the Lorentz boost transformation (5.68) to express it as follows:

$$i[\hat{x}',\hat{a}] = i[\hat{t}\sinh\phi + \hat{x}\cosh\phi,\hat{a}] \;\; > \;\; \theta \;\; (\forall\;\phi) \tag{5.101}$$

Now by choosing $\phi = -\psi$ and putting the condition in (5.91) we have the following lower bound for the Lorentzian distance:

$$d(\rho_0,\rho_z) > r = \sqrt{\theta}\sqrt{z^2+\bar{z}^2} \tag{5.102}$$

Now for evaluation of distance just like Euclidean Moyal plane, here too the pure states are represented by normalized density matrices . The distance between a pair of pure states (represented in terms of normalised Glauber-Sudarshan coherent state basis (2.12) as) $\rho_0 = |0\rangle\langle 0|$ and $\rho_z = |z\rangle\langle z|$ representing two time-like separated events $(0,0),(t_1,x_1)$ in forward light cone with $\rho_0 \prec \rho_z$, will be given by,

$$\begin{aligned} d(\rho_0,\rho_z) &= \inf_{\hat{a}\in\mathcal{B}}\{[\rho_z(\hat{a})-\rho_0(\hat{a})]^+\} = \inf_{\lambda}\{[\langle z|\hat{a}_\lambda|z\rangle - \langle 0|\hat{a}_\lambda|0\rangle]^+\} \\ &= \inf_{\lambda}\sqrt{\frac{\theta}{2}}\{[(z+\bar{z})\cosh\lambda - i(z-\bar{z})\sinh\lambda]^+\} \end{aligned} \tag{5.103}$$

where again we have defined $z = \frac{t_1+ix_1}{\sqrt{2\theta}}$ in the spirit of (5.20) in section-5.1 and made use of the form of $\hat{a}_\lambda$ given in (5.78). Now the infimum of the set is attained by minimising the set with respect to λ which immediately yields the desired distance between two pure states as,

$$d(\rho_z,\rho_w) = \sqrt{\theta}\sqrt{z^2+\bar{z}^2} = \sqrt{t_1^2-x_1^2}, \tag{5.104}$$

showing that (5.104) indeed agrees with the lower bound (5.102) .

Finally, note that the minimisation value of λ, for which the minimum value of the R.H.S of (5.103) is attained here is given by $\lambda = \tilde{\lambda}$, which satisfies $\tanh\tilde{\lambda} = \frac{i(z-\bar{z})}{z+\bar{z}}$. Now putting this value of $\lambda = \tilde{\lambda}$ in the expression of $\hat{a}_s$ (5.78) we have,

$$\hat{a}_s = r\cosh\psi[\hat{b}(\cosh\psi + i\sinh\psi) + \hat{b}^\dagger(\cosh\psi - i\sinh\psi)] \tag{5.105}$$

where we have used the same parametrization for z, as in (5.90). Now choosing $\psi = -\phi$ again, we have, using (5.68,5.69),

$$\hat{a}_s = r\cosh\phi\;(\hat{b}_\phi + \hat{b}_\phi^\dagger) = \sqrt{\frac{2}{\theta}}(r\cosh\phi)\,\hat{t}' \tag{5.106}$$

With the explicit form of $\hat{a}_s$, we can explicitly demonstrate that in a Lorentz-transformed frame (5.68), we have $[\hat{t}',\hat{a}_s] = 0$ and $i[\hat{x}',\hat{a}_s] = \theta\mathbf{1}$. Consequently, we can conclude that the saturation of the conditions (5.98) and (5.100) is achieved simultaneously with the infimum of the distance functional in a specific Lorentz frame, given our choice of $\hat{a}_s$ (5.78). Hence, we can confidently identify the right-hand side of (5.104) as the true Lorentzian distance between the pure states ρ_0 and ρ_z, with $\rho_0 \prec \rho_z$.

Since the distance is invariant under ISO(1,1) transformations (5.71), we can perform a suitable translation to obtain the distance between ρ_z and ρ_w where $\rho_z \prec \rho_w$. This is expressed as:

$$d(\rho_z, \rho_w) = \sqrt{\theta}\sqrt{(w-z)^2 + (\bar{w}-\bar{z})^2} = \sqrt{(t_2-t_1)^2 - (x_2-x_1)^2} \tag{5.107}$$

In this context, let us consider two pure states ρ_z and ρ_w representing the points (t_1, x_1) and (t_2, x_2) within the forward light cone. Similar to the Euclidean case, (5.107) also does not rely on the non-commutative parameter and mirrors the result of the geodesic distance in flat commutative 1+1-dimensional spacetime, which corresponds to a straight line.
Before concluding this section, we would like to emphasize an important observation. Just as in the Euclidean case, here too, we are compelled to consider the extended multiplier algebra that encompasses all the members of (5.78). This can be viewed as a minimalistic unitization of $\mathcal{A} = \mathcal{H}_q$, as discussed in [152, 156], which serves as the largest $\star$ sub-algebra of the aforementioned multiplier algebra where the coherent states remain finite. For our purposes, this minimalistic unitized subalgebra should suffice since the optimal element (5.78) belongs to it: $\rho_z(\hat{a}_\lambda) = \langle z|\hat{a}_\lambda|z\rangle < \infty$.
Additionally, it is worth noting that, as mentioned at the beginning of section-5.2, it can be shown from (5.39) that the fundamental symmetry $\mathcal{J}$ satisfies $[\mathcal{J}, a] = \sigma_1 \otimes [\mathbf{1}, a] = 0$ for all $a \in \tilde{\mathcal{A}}$. To verify the boundedness of the operator $[\mathcal{D}, \pi(a)]$, we can demonstrate using (5.78) that $\|[\mathcal{D}, \pi(a)]\|_{op} = \cosh\lambda < \infty$, which is clearly bounded for any pair of finitely separated states.

5.4 Chapter summary

In this chapter, we have demonstrated that the axiomatic formulation presented in [151, 154, 155] is well-suited for our purposes and allows us to compute the spectral distance between pairs of time-like separated pure states constructed using Glauber-Sudarshan coherent states on the Lorentzian Moyal plane. By employing un-normalized Bargmann-Fock coherent states, we have made the computation transparent, as the representation of $|.\rangle$ states and their dual $\langle.|$ states naturally emerges as anti-holomorphic and holomorphic functions, respectively, in our analysis involving Hilbert-Schmidt operators.

This exercise establishes confidence in the construction of the spectral triple for the 2D Lorentzian Moyal plane, as it successfully reproduces the expected results. To make further progress, it is necessary to replace the left module with a bi-module, which will allow for the incorporation of possible fluctuations in the Dirac operator. This, in turn, will enable the introduction of gauge and Higgs fields as prototypes. As a preliminary step, one could consider the doubled Lorentzian Moyal plane, akin to the doubled Euclidean Moyal plane discussed in [157, 162]. This approach could pave the way for realistic model building, where almost commutative spaces are upgraded to fully non-commutative spaces, offering insights into physics at the Planck scale.
It would also be intriguing to investigate whether there is any analogue of Pythagorean distance formula holds in the case of the doubled Lorentzian Moyal plane.

Chapter 6
Spectral triple on fuzzy sphere

Long ago, J. Madore introduced the concept of the fuzzy sphere (FS) S^2_* [164], which can be seen as a deformation of the conventional commutative 2-sphere, denoted as S^2. The fuzzy sphere serves as a simple example of NCG, where the operator valued coordinates satisfy $\mathfrak{su}(2)$ Lie algebra. In recent times, the high energy physics community has shown considerable interest in the fuzzy sphere due to its potential as a non-perturbative method in quantum field theory, as discussed in [165]. This method offers an alternative to lattice gauge theory, which discretizes spacetime into finite elements. One of the primary advantages of the fuzzy sphere is its ability to relate to both matrix models and NCG. The algebra of the fuzzy sphere can accommodate representations of the SU(2) Lie group, which are provided by M_{2n+1} matrices for the n-th representation.
The objective of this chapter is to construct a real structure on the fuzzy sphere, enabling us to endow it with real and even spectral data. This, in turn, will facilitate the construction of a simple toy model with SU(2) gauge symmetry, utilizing the fuzzy sphere S^2_* as the internal space within an almost commutative geometrical framework of Connes *et. al.* [67, 68], a plausible direction that can be explored in future.
While various proposals for Dirac operators on the fuzzy sphere have been put forth in [163, 164, 166, 167], along with the introduction of grading operators in some cases, the real structure on the fuzzy sphere remains a crucial missing component. In light of this, we consider the SU(2) covariant Dirac operator and grading operator on S^2_* as proposed by Watamura et al. in [69]. Our focus is on constructing the real structure on the fuzzy sphere within its lowest spin-1/2 representation. The fuzzy sphere in its 1/2 representation holds a special position within the family of fuzzy spheres. Not only does it represent a space of maximum noncommutativity, but the associated algebra $\mathcal{A}_F$ describing this space is given by the unital algebra of 2×2 matrices $M_2(\mathbb{C})$ is entirely spanned by the four generators: $(I, \vec{\sigma})$, which include the identity operator and the three $\mathfrak{su}(2)$ generators. It is important to note that this feature does not persist for S^2_* associated with higher representations ρ_n where $n \geq 1$. We have shown how this special feature can be utilized to construct the real structure. To achieve this, we need to extend the symmetry group of the fuzzy sphere from SO(3) to O(3), the total orthogonal group, which can eventually be assigned a KO dimension-4.
However, as we shall see that the Dirac operator violates the first-order condition, which is a necessary requirement in the formulation of the standard model of particle physics within the framework of almost commutative geometry, as discussed in [63, 143]. However, it was shown by Connes *et. al.* in [67, 68], that such a violation may actually be a "blessing in disguise" in the sense that, this formalism eventually enables the development of phenomenologically viable models such as the Pati-Salam model, which goes beyond the standard model. This provides further motivation for formulating a complete spectral data framework for the fuzzy sphere, which can be utilized to construct a toy model which may provide insights into some aspects of physics beyond the standard model.

This chapter is organized as follows:
In section-6.1, we present all the essential ingredients of the spectral triple for the fuzzy sphere, except for the real structure. This includes the algebra, the Hilbert space, the Dirac operator, and the grading operator for a spin-1/2 fuzzy sphere. The latter two structures are adopted from Watamura et al. [69]. In section-6.2, we derive the explicit forms of the eigenspinors of the Dirac operator as given in [69]. Furthermore, we construct the chiral spinors and explore the representation of the algebra and its opposite algebra in various Hilbert spaces. Moving on to section-6.3, we determine the KO dimension of the theory. Subsequently, we provide a detailed derivation of the real structure operator, using which the opposite algebra element, corresponding to an algebra element can be constructed and then we can verify its compliance with the zero-order condition. However, we find that the spectral data violates the first-order condition. Finally, in section-6.5, we draw our conclusions and discuss the future directions for this work.

6.1 Spectral triple for S^2_*

The coordinate algebra of S^2_* is given by

$$[\hat{x}_i, \hat{x}_j] = i\lambda\epsilon_{ijk}\hat{x}_k; \qquad i, j, k \in \{1, 2, 3\} \tag{6.1}$$

The parameter λ represents the NC parameter with the dimensions of length. The Casimir operator $r_n^2 = \rho_n(\hat{\vec{x}}^2) = \rho_n(\hat{x}_i).\rho_n(\hat{x}_i)$ permits only a discrete set of values, which can be expressed as $\lambda^2 n(n+1)$, where $n \in \frac{1}{2}, 1, \frac{3}{2}, ...$ corresponds to the SU(2) representation index and ρ_n denotes the n^{th} representation of the $\mathfrak{su}(2)$ generators.
In our study, we specifically focus on the $n = \frac{1}{2}$ representation, denoting it as $\hat{x}_i$. For higher-order representations $(n \geq 1, \frac{3}{2}, ...)$, we use the explicit notation $\rho_n(\hat{\vec{x}})$ when necessary. It is important to note that the infinite set of fuzzy spheres, each associated with a different radius, can be considered as foliation of the NC space R^3_*. Each of these fuzzy spheres can be identified with their respective radii $r_n = \lambda\sqrt{n(n+1)}$.
Now with $n = \frac{1}{2}$, a representation of the algebra (6.1) is furnished by following 2D Hilbert space

$$\mathcal{H}_c = span_{\mathbb{C}}\Big\{\Big|\frac{1}{2}\Big\rangle, \Big| -\frac{1}{2}\Big\rangle\Big\} \tag{6.2}$$

where the actions of $\hat{x}_i$ on $\mathcal{H}_c$ can be expressed in terms of ladder operators $\hat{x}\pm$ and $\hat{x}_3$ as follows:

$$\begin{aligned}
&\hat{x}_+\Big|\frac{1}{2}\Big\rangle = \hat{x}_-\Big| -\frac{1}{2}\Big\rangle = 0; \quad \hat{x}_+\Big| -\frac{1}{2}\Big\rangle = \lambda\Big|\frac{1}{2}\Big\rangle; \hat{x}_-\Big|\frac{1}{2}\Big\rangle = \lambda\Big| -\frac{1}{2}\Big\rangle \\
&\hat{x}_3\Big| \pm\frac{1}{2}\Big\rangle = \pm\frac{\lambda}{2}\Big| \pm\frac{1}{2}\Big\rangle; \quad \hat{x}_\pm = \hat{x}_1 \pm i\hat{x}_2
\end{aligned} \tag{6.3}$$

The orthonormality and completeness condition for $\mathcal{H}_c$ are given as,

$$\langle m|n\rangle = \delta_{mn}; \quad \Big|\frac{1}{2}\Big\rangle\Big\langle\frac{1}{2}\Big| + \Big| -\frac{1}{2}\Big\rangle\Big\langle -\frac{1}{2}\Big| := \mathbf{I}_{\mathcal{H}_c} \in \mathcal{H}_q \;\; ; \quad m, n \in \{\frac{1}{2}, -\frac{1}{2}\} \tag{6.4}$$

where $\mathcal{H}_q$ is defined below in (6.5).
As for the algebra of the triplet, we consider $\mathcal{A}_F = \mathcal{H}_q$, which corresponds to the space of Hilbert-

Schmidt (HS) operators that act on $\mathcal{H}_c$ given by,

$$\mathcal{A}_F := \mathcal{H}_q = \mathcal{H}_c \otimes \tilde{\mathcal{H}_c} = span_{\mathbb{C}}\Big\{\Big|\frac{1}{2}\Big\rangle\Big\langle\frac{1}{2}\Big|, \quad \Big|\frac{1}{2}\Big\rangle\Big\langle -\frac{1}{2}\Big|, \quad \Big| -\frac{1}{2}\Big\rangle\Big\langle\frac{1}{2}\Big|, \quad \Big| -\frac{1}{2}\Big\rangle\Big\langle -\frac{1}{2}\Big|\Big\} \tag{6.5}$$

where $\tilde{\mathcal{H}_c}$ is the dual of $\mathcal{H}_c$. These four basis elements of the form $|m\rangle\langle n| := |m,n)$ also satisfies orthonormality and completeness as (6.4),

$$(m',n'|m,n) = \delta_{m,m'}\delta_{n,n'}; \quad \sum_{m,n} |m,n)(m,n| = \mathbf{I}_{\mathcal{H}_q} \in \mathcal{H}_q \otimes \tilde{\mathcal{H}_q} \tag{6.6}$$

where

$$(m',n'| = |m',n')^* = (|m'\rangle\langle n'|)^* = |n'\rangle\langle m'| \tag{6.7}$$

The inner product between a pair of HS operators $|\psi),|\phi) \in \mathcal{H}_q$ is defined as ,

$$(\phi|\psi) := tr_{\mathcal{H}_c}(\phi^*\psi) = \sum_{i=-\frac{1}{2}}^{+\frac{1}{2}} \langle i|\phi^*\psi|i\rangle; \quad |i\rangle \in \mathcal{H}_c \tag{6.8}$$

The $*$-operation in the algebra $\mathcal{A}_F = \mathcal{H}_q$, which corresponds to the involution operation, can be defined as the simple Hermitian conjugation in this case. It satisfies the following properties:

$$(a^*)^* = a; \quad (ab)^* = b^*a^* \quad \forall a,b \in \mathcal{A}_F \tag{6.9}$$

The inner automorphism symmetry of $\mathcal{A}_F$ can be associated with the gauge symmetry of $\mathcal{A}_F$. As for the Hilbert space of the spectral triple, which should function as a bi-module of the algebra, we consider it to be $\mathbb{C}^2 \otimes \mathcal{H}_q$ i.e.

$$\mathcal{H}_F := \mathbb{C}^2 \otimes \mathcal{H}_q = span_{\mathbb{C}}\{|\phi_1)), |\phi_2)), |\phi_3)), |\phi_4)), |\phi_5)), |\phi_6)), |\phi_7)), |\phi_8))\} \tag{6.10}$$

where

$$|\phi_1)) = \begin{pmatrix} \big|\frac{1}{2}\big\rangle\big\langle\frac{1}{2}\big| \\ 0 \end{pmatrix}, \quad |\phi_2)) = \begin{pmatrix} \big|\frac{1}{2}\big\rangle\big\langle -\frac{1}{2}\big| \\ 0 \end{pmatrix}, \quad |\phi_3)) = \begin{pmatrix} \big| -\frac{1}{2}\big\rangle\big\langle\frac{1}{2}\big| \\ 0 \end{pmatrix}, \quad |\phi_4)) = \begin{pmatrix} \big| -\frac{1}{2}\big\rangle\big\langle -\frac{1}{2}\big| \\ 0 \end{pmatrix}$$
$$|\phi_5)) = \begin{pmatrix} 0 \\ \big|\frac{1}{2}\big\rangle\big\langle\frac{1}{2}\big| \end{pmatrix}, \quad |\phi_6)) = \begin{pmatrix} 0 \\ \big|\frac{1}{2}\big\rangle\big\langle -\frac{1}{2}\big| \end{pmatrix}, \quad |\phi_7)) = \begin{pmatrix} 0 \\ \big| -\frac{1}{2}\big\rangle\big\langle\frac{1}{2}\big| \end{pmatrix}, \quad |\phi_8)) = \begin{pmatrix} 0 \\ \big| -\frac{1}{2}\big\rangle\big\langle -\frac{1}{2}\big| \end{pmatrix} \tag{6.11}$$

where the $\{|\phi_\mu)); \mu = 1,2,...,8\}$ furnish a complete and orthonormal basis for $\mathbb{C}^2 \otimes \mathcal{H}_q$:

$$((\phi_\mu|\phi_{\mu'})) = \delta_{\mu\mu'}; \qquad \sum_{\mu=1}^{8} |\phi_\mu))\,((\phi_\mu| = \mathbf{I}_{\mathcal{H}_F} \in \mathcal{H}_F \otimes \tilde{\mathcal{H}}_F \tag{6.12}$$

Here the inner-product between a pair of elements $|\psi_1)), |\psi_2)) \in \mathcal{H}_F$ has been defined as,

$$((\psi_1|\psi_2)) = tr_{\mathbb{C}^2\otimes\mathcal{H}_c}(\psi_1^\dagger\psi_2) \tag{6.13}$$

where we have reserved the dagger (†) symbol for its use in $\mathcal{H}_F$:

$$((\psi| := |\psi))^\dagger = \begin{pmatrix} (\xi_1| & (\xi_2| \end{pmatrix} \text{ for } |\psi)) := \begin{pmatrix} |\xi_1) \\ |\xi_2) \end{pmatrix}; \quad |\xi_1), |\xi_2) \in \mathcal{H}_q, \text{ and } (\xi_1|, (\xi_2| \in \tilde{\mathcal{H}}_q \tag{6.14}$$

We shall use the term "canonical basis" to refer to the basis $|\phi_\lambda))$ where $\lambda = 1, 2, ..., 8$. It is important to note that we have a hierarchy of Hilbert spaces: $\mathcal{H}_c$, $\mathcal{H}_q = \mathcal{H}_c \otimes \tilde{\mathcal{H}}_c$, and $\mathcal{H}_F = \mathbb{C}^2 \otimes \mathcal{H}_q$. The corresponding elements in these spaces are denoted by $|.\rangle$, $|.)$, and $|.))$ respectively. Additionally, we can consider the Hilbert space $\mathcal{H}_F$ as a module (or more specifically, a bi-module since both left and right actions can be defined) of the algebra $\mathcal{A}_F$, utilizing the diagonal representation $\pi(a) = \text{diag}(a, a)$ for all $a \in \mathcal{A}_F$.
Finally, the Dirac operator and the chirality operator can be written as [69],

$$D_F = \frac{i}{\lambda r_n} \gamma_F \epsilon_{ijk} \sigma_i \hat{x}_j{}^R \hat{x}_k; \quad \gamma_F = \frac{1}{\mathcal{N}} (\vec{\sigma}.\hat{\vec{x}}^R - \frac{\lambda}{2}) \tag{6.15}$$

In our notation, the superscript R denotes the right action of the operator $\hat{x}_j$. In absence of any superscript, it implies that the action of $\hat{x}_i$ is assumed to be from the left by default, and thus the superscript L is suppressed in this case.
While our analysis focuses on the $n = \frac{1}{2}$ representation, it is worth noting that for the general case, r_n represents the radius of the n-th fuzzy sphere as indicated below (6.1). Furthermore, $\mathcal{N} = \lambda(n + \frac{1}{2})$ serves as a normalization constant for γ_F in (6.15) ensuring that the condition $\gamma_F^2 = 1$ holds. For the specific case of $n = \frac{1}{2}$, these expressions simplify to the following forms:

$$r_{\frac{1}{2}}^2 = \frac{3\lambda}{4} \text{ and } \mathcal{N} = \lambda \tag{6.16}$$

The algebra $\mathcal{A}_F$ (6.5), the Hilbert space $\mathcal{H}_F$ (6.10), and the Dirac operator D_F (6.15) and the grading operator γ_F (6.15) form the essential components for an *even* spectral triple for the fuzzy sphere. In addition to these data , to construct a *real* spectral triple, we also need a real structure $\mathcal{J}_F$ the construction of which is the main objective of section-6.4. It should be recalled in this context that in order to formulate a gauge theory in the framework of an almost commutative manifold, we can therefore compose these spectral data with those of a compact Euclidean manifold M_4 to get the composite spectral triple as

$$\mathcal{A} = C^\infty(M_4, \mathcal{A}_F); \quad \mathcal{H} = L^2(M_4) \otimes \mathcal{H}_F; \quad D = D_M \otimes \mathbf{I} + \gamma_5 \otimes D_F; \quad \gamma = \gamma_5 \otimes \gamma_F; \quad \mathcal{J} = \mathcal{J}_M \otimes \mathcal{J}_F$$

where $\Big(C^\infty(M_4), L^2(M_4), D_M, \gamma_5, \mathcal{J}_M\Big)$ are the spectral data for the Euclidean manifold. $\mathcal{J}_M$ is the real structure for M_4 and its action is simply given by a complex conjugation. Thus the only missing link in this above construction is the real structure $\mathcal{J}_F$ and the objective of this chapter is precisely this. We now embark on this task by determining the eigen-spinors of $\mathcal{D}_F$ in the following sub-section.

6.2 Determination of eigenspinors of $\mathcal{D}_F$ and the chiral basis

To obtain $\mathcal{J}_F$, we begin by determining the eigen-spinors of the Dirac operator D_F (6.15). Since the Hilbert space $\mathbb{C}^2 \otimes \mathcal{H}_q$ is 8-dimensional, we expect to have 8 linearly independent eigen-spinors. Like $|\phi_\lambda))$, these eigen-spinors should also form a complete orthonormal basis for $\mathbb{C}^2 \otimes \mathcal{H}_q$. To accomplish this, we make the following ansatz:

$$|\psi)) = \begin{pmatrix} \sum_{m,n} A_{m,n}|m\rangle\langle n| \\ \sum_{p,q} B_{p,q}|p\rangle\langle q| \end{pmatrix} \in \mathbb{C}^2 \otimes \mathcal{H}_q; \quad n,m,p,q \in \left\{\frac{1}{2}, -\frac{1}{2}\right\} \tag{6.17}$$

satisfying the eigenvalue equation

$$D_F|\psi)) = m|\psi)) \tag{6.18}$$

and solve it. Also note that $[D_F, \hat{\vec{J}}] = 0$, where $\hat{\vec{J}} = \{J_1, J_2, J_3\}$ are the 3-components of the total angular momentum

$$\hat{J}_i = \hat{L}_i + \frac{\sigma_i}{2}; \;\; L_i = \frac{1}{\lambda}(\hat{x}_i - \hat{x}_i^R) \tag{6.19}$$

where $\vec{L}$ can be regarded as the orbital angular momentum and like $\vec{J}$ this too is dimensionless. At this stage, it becomes evident that these eigen-spinors can be labeled by their respective 'mass' m (Dirac eigen value) (6.18), as well as the eigenvalues of $\vec{J}^2$ and J_3. We can therefore denote these eigen-spinors as $|m; j, j_3))$, where they satisfy the following conditions:

$$D_F|m;j,j_3)) = m|m;j,j_3)); \quad \hat{J}_3|m;j,j_3)) = j_3|m;j,j_3)); \quad \hat{\vec{J}}^2|m;j,j_3)) = j(j+1)|m;j,j_3)) \tag{6.20}$$

A long but straight forward calculation yields the following structures for the normalised eigen spinors,

$$|\psi_1)) = \frac{1}{3-\sqrt{3}} \begin{pmatrix} \left|\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| + (\sqrt{3}-1)\left|-\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| \\ (2-\sqrt{3})\left|\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| \end{pmatrix} = \left|1; \frac{1}{2}, \frac{1}{2}\right))$$

$$|\psi_2)) = \frac{1}{3-\sqrt{3}} \begin{pmatrix} (2-\sqrt{3})\left|-\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| \\ (\sqrt{3}-1)\left|\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| + \left|-\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| \end{pmatrix} = \left|1; \frac{1}{2}, -\frac{1}{2}\right))$$

$$|\psi_3)) = \frac{1}{3-\sqrt{3}} \begin{pmatrix} (1-\sqrt{3})\left|-\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| + (2-\sqrt{3})\left|\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| \\ \left|\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| \end{pmatrix} = \left|-1; \frac{1}{2}, \frac{1}{2}\right))$$

$$|\psi_4)) = \frac{1}{3-\sqrt{3}} \begin{pmatrix} \left|-\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| \\ (2-\sqrt{3})\left|-\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| + (1-\sqrt{3})\left|\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| \end{pmatrix} = \left|-1; \frac{1}{2}, -\frac{1}{2}\right))$$

$$|\psi_5)) = \begin{pmatrix} \left|\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| \\ 0 \end{pmatrix} = \left|0; \frac{3}{2}, \frac{3}{2}\right)), \qquad |\psi_6)) = \begin{pmatrix} 0 \\ \left|-\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| \end{pmatrix} = \left|0; \frac{3}{2}, -\frac{3}{2}\right))$$

$$|\psi_7)) = \frac{1}{\sqrt{3}} \begin{pmatrix} -\left|\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| + \left|-\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| \\ \left|\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| \end{pmatrix} = \left|0; \frac{3}{2}, \frac{1}{2}\right)), \quad |\psi_8)) = \frac{1}{\sqrt{3}} \begin{pmatrix} \left|-\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| \\ \left|\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| - \left|-\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| \end{pmatrix} = \left|0; \frac{3}{2}, -\frac{1}{2}\right)) \tag{6.21}$$

As mentioned earlier, these $|\psi_\lambda))$'s ($\lambda \in 1; 2, ..., 8$) also satisfy the orthonormality and completeness relations, just like the $|\phi_\lambda))$'s (6.12).

An important observation is that both eigenvalues $m = \pm 1$ exhibit double degeneracy, representing spin-$\frac{1}{2}$ doublets. In other words, the pairs of doublets (ψ_1, ψ_2) and (ψ_3, ψ_4) satisfy the following relations:

$$\begin{aligned} &\hat{J}_+|\psi_1)) = \hat{J}_-|\psi_2)) = 0; \quad \hat{J}_+|\psi_2)) = |\psi_1)); \quad \hat{J}_-|\psi_1)) = |\psi_2)) \\ &\hat{J}_+|\psi_3)) = \hat{J}_-|\psi_4)) = 0; \quad \hat{J}_+|\psi_4)) = |\psi_3)); \quad \hat{J}_-|\psi_3)) = |\psi_4)); \quad \hat{J}_\pm = \hat{J}_1 \pm i\hat{J}_2 \end{aligned} \tag{6.22}$$

Similarly, the massless spinors ($m = 0$) exhibit a degeneracy of 4, forming $j = \frac{3}{2}$ quadruplet. In this case, the highest weight state $|\psi_5))$ is annihilated by $\hat{J}_+$, while the lowest weight state $|\psi_6))$ is annihilated by $\hat{J}_-$. All four states in this quadruplet can be connected through multi-fold applications of these ladder operators $\hat{J}_\pm$.
Furthermore , the chirality operator γ_F can be shown to relate massive spinors as,

$$\psi_3 = \gamma_F\psi_1; \quad \psi_4 = \gamma_F\psi_2 \tag{6.23}$$

so that the condition $\{\gamma_F, D_F\} = 0$ is trivially satisfied. This enables us to introduce the chiral basis as

$$\chi_1^+ := \frac{1}{\sqrt{2}}(\psi_1 + \psi_3) = \frac{1}{\sqrt{2}}(1 + \gamma_F)\psi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \left|\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| \\ \left|\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| \end{pmatrix} = \left|1; \frac{1}{2}, \frac{1}{2}\right)\Big)$$

$$\chi_1^- := \frac{1}{\sqrt{2}}(\psi_1 - \psi_3) = \frac{1}{\sqrt{2}}(1 - \gamma_F)\psi_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} \left|\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| + 2\left|-\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| \\ -\left|\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| \end{pmatrix} = \left|-1; \frac{1}{2}, \frac{1}{2}\right)\Big)$$

$$\chi_2^+ := \frac{1}{\sqrt{2}}(\psi_2 + \psi_4) = \frac{1}{\sqrt{2}}(1 + \gamma_F)\psi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} \left|-\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| \\ \left|-\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| \end{pmatrix} = \left|1; \frac{1}{2}, -\frac{1}{2}\right)\Big)$$

$$\chi_2^- := \frac{1}{\sqrt{2}}(\psi_2 - \psi_4) = \frac{1}{\sqrt{2}}(1 - \gamma_F)\psi_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} -\left|-\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| \\ \left|-\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| + 2\left|\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| \end{pmatrix} = \left|-1; \frac{1}{2}, -\frac{1}{2}\right)\Big) \tag{6.24}$$

Note that in this case, we have employed the notation $|\gamma; j, j_3))$, where γ represents the eigenvalue of the chirality operator γ_F, to label the states. It satisfies the equation $\gamma_F|\gamma; j, j_3)) = \gamma|\gamma; j, j_3))$.
Furthermore, it is important to note that all the zero modes $|\psi_5)), ..., |\psi_8))$ possess a negative chirality value, specifically $\gamma = -1$. Consequently, we can express them as follows:

$$|\psi_5)) = \begin{pmatrix} \left|\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| \\ 0 \end{pmatrix} = \left|-1, \frac{3}{2}, \frac{3}{2}\right)\Big), \qquad |\psi_6)) = \begin{pmatrix} 0 \\ \left|-\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| \end{pmatrix} = \left|-1, \frac{3}{2}, -\frac{3}{2}\right)\Big)$$

$$|\psi_7)) = \frac{1}{\sqrt{3}} \begin{pmatrix} -\Big|\frac{1}{2}\Big\rangle\Big\langle\frac{1}{2}\Big| + \Big|-\frac{1}{2}\Big\rangle\Big\langle-\frac{1}{2}\Big| \\ \Big|\frac{1}{2}\Big\rangle\Big\langle-\frac{1}{2}\Big| \end{pmatrix} = \Big|-1, \frac{3}{2}, \frac{1}{2}\Big)\Big)$$

$$|\psi_8)) = \frac{1}{\sqrt{3}} \begin{pmatrix} \Big|-\frac{1}{2}\Big\rangle\Big\langle\frac{1}{2}\Big| \\ \Big|\frac{1}{2}\Big\rangle\Big\langle\frac{1}{2}\Big| - \Big|-\frac{1}{2}\Big\rangle\Big\langle-\frac{1}{2}\Big| \end{pmatrix} = \Big|-1, \frac{3}{2}, -\frac{1}{2}\Big)\Big) \tag{6.25}$$

Let us take a moment to discuss the physical interpretation behind the assignment of different j_3 eigenvalues to the Dirac and chiral spinors. At this juncture, it is important to point that the $(2n+1)$-dimensional Hilbert space $\mathcal{H}_c$ for an n-th fuzzy sphere can be identified as a finite-dimensional subspace within an infinite-dimensional Fock space generated by the ladder operators $(\hat{a}_1, \hat{a}_1^\dagger)$ and $(\hat{a}_2, \hat{a}_2^\dagger)$, which satisfy the commutation relation $[\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha\beta}$. This Fock space represents the Hilbert space of 2D decoupled harmonic oscillators, given by:

$$\mathcal{F} := Span_{\mathbb{C}}\{|n_1\rangle \otimes |n_2\rangle \equiv |n_1, n_2\rangle = \frac{(\hat{a}_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(\hat{a}_2^\dagger)^{n_2}}{\sqrt{n_2!}} |0,0\rangle \;\; \forall \;\; n_1, n_2 \in \mathbb{Z}\} = \oplus_{n \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}} \mathcal{F}_n \tag{6.26}$$

It is important to note that $\mathcal{F}$ can be decomposed into a direct sum of such subspaces $\mathcal{F}_n$, each of which provides an irreducible representation of the $\mathfrak{su}(2)$ Lie algebra, for a fixed value of n, spanned by the states $|n_1, n_2\rangle$ satisfying $\frac{n_1+n_2}{2} = n$.

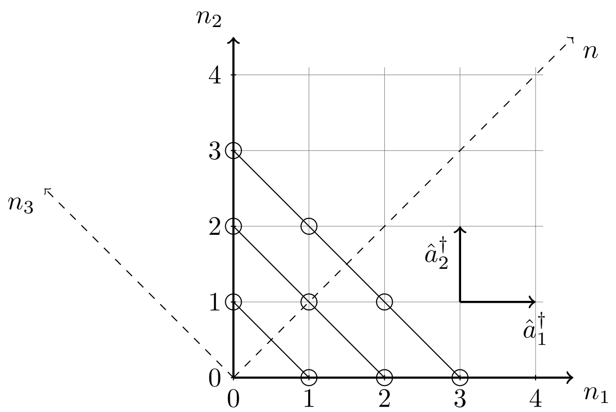

Figure 6.1: Graphical representation of $\mathfrak{su}(2)$ representation spaces

In the above diagram, the x axis represents the Hilbert space spanned by the basis states $\{|n_1, 0\rangle = \frac{(\hat{a}_1 \dagger)^{n_1}}{\sqrt{n_1!}}|0,0\rangle\}$ which are represented by the set of discrete points $\{(0,0),(1,0),(2,0),...(n_1,0),...\}$. Like-wise , the y axis represents the Hilbert space spanned by the basis states $\{|0, n_2\rangle = \frac{(\hat{a}_2^\dagger)^{n_2}}{\sqrt{n_2!}}|0,0\rangle\}$ and are in correspondence with the discrete set of points $\{(0,0),(0,1),(0,2),...,(0,n_2),..\}$. Note that the state $|0,0\rangle$ is common to both the Hilbert spaces and is annihilated by both $\hat{a}_1$ and $\hat{a}_2$: $\hat{a}_1|0,0\rangle = \hat{a}_2|0,0\rangle = 0$.
Now it can be seen that the lines joining (1,0) to (0,1) , (2,0) to (0,2) and in general (n,0) to (0,n) have fixed values of n, and contains $2n+1$ number of distinct points whose coordinates are integer-valued and represent individual basis elements $|n_1, n_2\rangle$ with $n_1 + n_2 = 2n$ and thus spans $(2n+1)$ dimensional subspace $\mathcal{F}_n$ of $\mathcal{F}$ (6.26). Clearly, such states can alternatively be lebelled by n and n_3

as $|n_1,n_2\rangle = |n,n_3\rangle$. Again for $n=1/2$, in particular, we can write $\mathcal{F}_{\frac{1}{2}} = span\{|1,0\rangle,|0,1\rangle\} = span\Big\{\Big|\frac{1}{2},\frac{1}{2}\Big\rangle,\Big|\frac{1}{2},-\frac{1}{2}\Big\rangle\Big\} = \mathcal{H}_c$ (6.2). Clearly, all these subspaces , which furnish, by themselves the irreducible representation of $\mathfrak{su}(2)$ Lie algebra have a fixed value of the Casimir operator. In fact, one can check that although the operators $\hat{a}_i/\hat{a}_i^\dagger$ do not preserve these sub-spaces on their own, a certain bi-linear expressions e.g. $\hat{a}_1^\dagger\hat{a}_2$ or $\hat{a}_2^\dagger\hat{a}_1$ act as ladder operators within these respective sub-spaces. To see this consider any basis element $|n_1,n_2\rangle = |n,n_3\rangle \in \mathcal{F}_n$ and one can easily see from Fig. 6.1 that $\hat{a}_1^\dagger\hat{a}_2|n,n_3\rangle \sim |n,n_3-1\rangle$ and $\hat{a}_1\hat{a}_2^\dagger|n,n_3\rangle \sim |n,n_3+1\rangle$. They can also be combined to give hermitian operators as [169],

$$\rho_n(\hat{\vec{x}}) = \frac{\lambda}{2}\hat{\xi}^\dagger\vec{\sigma}\hat{\xi}\ ; \qquad \hat{\xi} = \begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix} \tag{6.27}$$

which correspond to $\mathfrak{su}(2)$ generators in n^{th} representation, fulfilling (6.1) and the above map is called called Jordan-Schwinger (JS) map. It can be demonstrated that the subspace $\mathcal{F}_n$ remains invariant under the action of the ladder operators $\rho_n(\hat{x}_\pm)$ defined in (6.3,6.27). Furthermore, the Casimir operator associated with the n^{th} representation satisfies the following relation:

$$\rho_n(\hat{\vec{x}}^2) = \rho_n(\hat{\vec{x}}).\rho_n(\hat{\vec{x}}) = \lambda^2\, n\Big(n+1\Big) \tag{6.28}$$

and can be assigned as the value of the square of the radius r_n for the associated S_*^2 (see below (6.1)). However, it should be noted that in our construction, we are working with the Hilbert space $\mathcal{H}_F = \mathbb{C}^2\otimes\mathcal{H}_q$ (see (6.10)), where $\mathcal{H}_q = \mathcal{H}_c\otimes\tilde{\mathcal{H}}_c$. By utilizing the Clebsch-Gordan decomposition of the SU(2) representation, $\mathcal{H}_q$ can be decomposed into a direct sum of a 3-dimensional representation (triplet $\mathcal{F}_1$) and a 1-dimensional representation (singlet $\mathcal{F}_0$) of $\mathfrak{su}(2)$, given by $2\otimes 2 = 3\oplus 1$, as follows (see Fig. 6.2):

$$\begin{aligned}
&\text{Singlet} \quad \rightarrow \quad |\xi) := \frac{1}{\sqrt{2}}\Big[\Big|\frac{1}{2}\Big\rangle\Big\langle\frac{1}{2}\Big| + \Big|-\frac{1}{2}\Big\rangle\Big\langle-\frac{1}{2}\Big|\Big] \equiv |0,0)\\
&\text{Triplet} \quad \rightarrow \quad \begin{cases} |\xi_{-1}) := \Big|-\frac{1}{2}\Big\rangle\Big\langle\frac{1}{2}\Big| & \equiv |1,-1)\\ |\xi_0) := \frac{1}{\sqrt{2}}\Big[\Big|\frac{1}{2}\Big\rangle\Big\langle\frac{1}{2}\Big| - \Big|-\frac{1}{2}\Big\rangle\Big\langle-\frac{1}{2}\Big|\Big] & \equiv |1,0)\\ |\xi_1) := -\Big|\frac{1}{2}\Big\rangle\Big\langle-\frac{1}{2}\Big| & \equiv |1,1)\end{cases}
\end{aligned} \tag{6.29}$$

where $|.,.)$ gives the (n,n_3) values.Clearly, the eigenvalue r_n^2 in the triplet subspace can be interpreted as the square of the radius for the $n=1$ fuzzy S_*^2 in units of λ^2, given by $r_1^2 = 1(1+1) = 2$.
Similarly, the Hilbert space $\mathcal{H}_F = \mathbb{C}^2\otimes\mathcal{H}_q$, spanned by the Dirac spinors (or alternatively, the chiral spinors), can be shown to split into a quadruplet and a pair of doublet states using the Clebsch-Gordan rule. This decomposition can be expressed as $2\times(2\times 2) = 4\oplus 2\oplus 2$, and it spans the representation space of the spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ fuzzy spheres, with $r(\frac{3}{2})^2 = \frac{3}{2}(\frac{3}{2}+1)\lambda^2$ and $r(\frac{1}{2})^2 = \frac{1}{2}(\frac{1}{2}+1)\lambda^2$, respectively.
Therefore, the massive Dirac spinors, which form the pair of doublets, are assigned $j_3 = \pm\frac{1}{2}$ values, while the 4 massless spinors belonging to the quadruplet are assigned j_3 values of $-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2}$ (refer to (6.21) and (6.22) for more details).

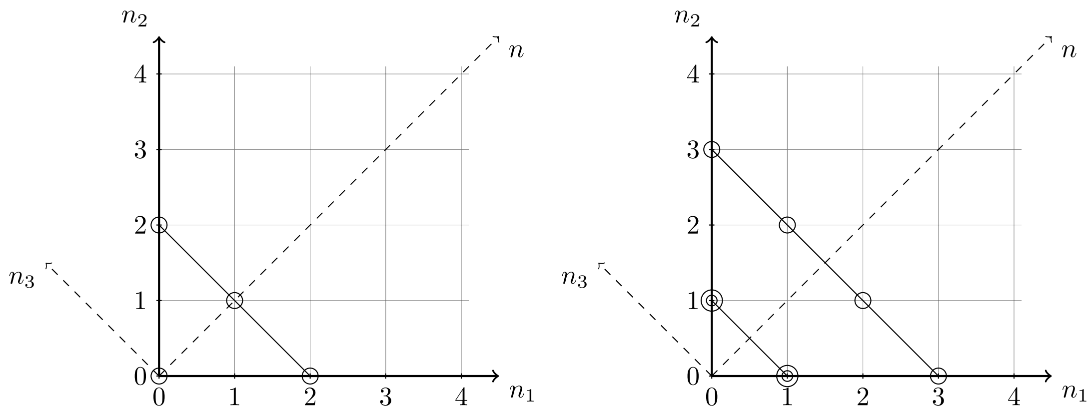

Figure 6.2: Graphical representation of $\mathcal{H}_q$ and $\mathcal{H}_F$ respectively

6.3 Representation of algebra generators in $\mathcal{H}_c, \mathcal{H}_q$, and $\mathcal{H}_F$

Before embarking into the computation of the real structure, it will be advantageous to begin by constructing the representation of algebra generators in the Hilbert spaces $\mathcal{H}_c, \mathcal{H}_q$, and $\mathcal{H}_F$.
To get the representation of the algebra in the Hilbert space $\mathcal{H}_c$, we can left multiply the identity $I_{\mathcal{H}_c}$ (6.4) by $\hat{x}_1$ and utilize (6.3) to obtain the following expression,

$$\hat{x}_1 = \hat{x}_1 I_{\mathcal{H}_c} = \hat{x}_1 \left(\left|\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| + \left|-\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| \right) = \frac{1}{2}(\hat{x}_+ + \hat{x}_-) \left(\left|\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| + \left|-\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| \right)$$
$$= \frac{\lambda}{2} \left(\left|\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| + \left|-\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| \right) = \frac{\lambda}{2}\sigma_1 \qquad (6.30)$$

We can proceed similarly to identify $\hat{x}_2$ and $\hat{x}_3$ as follows, where the rows and columns of the Pauli matrices σ_2 and σ_3 are labeled by $\left(\frac{1}{2}\right)$ and $\left(-\frac{1}{2}\right)$ respectively.

$$\hat{x}_2 = \hat{x}_2 I_{\mathcal{H}_c} = \frac{-i\lambda}{2} \left(\left|\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| - \left|-\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| \right) = \frac{\lambda}{2}\sigma_2;$$
$$\hat{x}_3 = \hat{x}_3 I_{\mathcal{H}_c} = \frac{\lambda}{2} \left(\left|\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| - \left|-\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| \right) = \frac{\lambda}{2}\sigma_3 \qquad (6.31)$$

so that we can combine all these three components to write compactly as.

$$\hat{\vec{x}} = \frac{\lambda}{2}\vec{\sigma} \qquad (6.32)$$

These are the representations of the left actions of the coordinate operators $\hat{x}_i$ on $\mathcal{H}_c$. Similarly, the corresponding expressions for the right actions of $\hat{x}_i^R$ on $\tilde{\mathcal{H}}_c$ can be obtained (since the action of $\hat{x}_i^R$ is only defined on $\tilde{\mathcal{H}}_c$) as follows:

$$\hat{\vec{x}}^R = \hat{\vec{x}}^R I_{\mathcal{H}_c} = \left(\left|\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| + \left|-\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right| \right) \hat{\vec{x}} = \frac{\lambda}{2}\vec{\sigma} \qquad (6.33)$$

Note that both the left and right actions on $\mathcal{H}_c$ and $\tilde{\mathcal{H}}_c$ have the same forms in terms of Pauli matrices (6.32,6.33), but their domains of action are different. The left action is defined on $\mathcal{H}_c$ for $\hat{x}_i$, whereas the right action is defined on $\tilde{\mathcal{H}}_c$ for $\hat{x}_i^R$. It is evident that $\hat{\vec{x}}^R$ satisfies the condition of being the opposite algebra i.e.

$$[\hat{x}_i, \hat{x}_j^R] = 0; \quad (x_i x_j)^R = x_j^R x_i^R \tag{6.34}$$

and therefore the Lie- algebra satisfied by $\hat{x}_i^R$ picks up a minus sign in its structure constant:

$$[\hat{x}_i^R, \hat{x}_j^R] = -i\lambda\epsilon_{ijk}\hat{x}_k^R \tag{6.35}$$

Now we will represent the algebra generators in the basis of $\mathcal{H}_q$. To do this, let us denote the four basis elements of $\mathcal{H}_q$ in (6.5) as,

$$\begin{aligned} |\eta_1) &= \Big|\frac{1}{2}\Big\rangle\Big\langle\frac{1}{2}\Big| = \Big|\frac{1}{2}, \frac{1}{2}\Big); \quad |\eta_2) = \Big|\frac{1}{2}\Big\rangle\Big\langle -\frac{1}{2}\Big| = \Big|\frac{1}{2}, -\frac{1}{2}\Big); \\ |\eta_3) &= \Big|-\frac{1}{2}\Big\rangle\Big\langle\frac{1}{2}\Big| = \Big|-\frac{1}{2}, \frac{1}{2}\Big); \quad |\eta_4) = \Big|-\frac{1}{2}\Big\rangle\Big\langle -\frac{1}{2}\Big| = \Big|-\frac{1}{2}, -\frac{1}{2}\Big) \end{aligned} \tag{6.36}$$

so that the completeness relations (6.6) for $\mathcal{H}_q$ can be written as

$$\sum_{\mu=1}^{4} |\eta_\mu)(\eta_\mu| = \mathbf{I}_{\mathcal{H}_q}; \quad \mu = 1,2,3,4 \tag{6.37}$$

To utilize them in our further computations, it will be advantageous to write the basis $|\eta_\mu)$ in terms of the algebra generators at this point.

$$\begin{aligned} |\eta_1) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{\mathbf{1}}{2} + \frac{\hat{x}_3}{\lambda}; \qquad |\eta_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{\hat{x}_+}{\lambda}; \\ |\eta_3) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{\hat{x}_-}{\lambda}; \quad |\eta_4) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\mathbf{1}}{2} - \frac{\hat{x}_3}{\lambda} \end{aligned} \tag{6.38}$$

As we did in (6.30), we can now identify the representation of $\hat{x}_i$ on $\mathcal{H}_q$ by simply multiplying $\mathbf{I}_{\mathcal{H}_q}$ (6.37) by $\hat{x}_i$ to obtain the result after a simple calculation,

$$\hat{x}_j = \hat{x}_j I_{\mathcal{H}_q} = \sum_{\mu=1}^{4} |\hat{x}_j \eta_\mu)(\eta_\mu| = \frac{\lambda}{2}\sigma_j \otimes I_2 \tag{6.39}$$

And for the right action $\hat{x}_j^R$ we can like-wise obtain

$$\hat{x}_j^R = \hat{x}_j^R I_{\mathcal{H}_q} = \sum_{\mu=1}^{4} |\eta_\mu \hat{x}_j)(\eta_\mu| = \frac{\lambda}{2} I_2 \otimes \sigma_j \tag{6.40}$$

so that we can write both (6.39,6.40) in a more compact index-free notation as

$$\hat{\vec{x}} = \frac{\lambda}{2}\vec{\sigma} \otimes I_2; \quad \hat{\vec{x}}^R = \frac{\lambda}{2} I_2 \otimes \vec{\sigma} \tag{6.41}$$

Note that in (6.39, 6.40), the operator $\hat{x}_i$ acts on $|\eta_\mu)$ from the left and right, respectively, without affecting the $(\eta_\mu|$ sector. Additionally, the operators (6.39, 6.40) act on $\mathcal{H}_q$, where the first entry of

the tensor product acts on $\mathcal{H}_q$ from the left and the second entry acts on $\mathcal{H}_q$ from the right. It can be easily verified that both $\hat{\vec{x}}$ and $\hat{\vec{x}}^R$ satisfy their respective $\mathfrak{su}(2)$ and opposite $\mathfrak{su}(2)^R$ algebras. These algebras are isomorphic to (6.1) and (6.35), respectively.
Finally, we need to determine the representation of $\hat{x}_i$ on $\mathcal{H}_F = \mathbb{C}^2 \otimes \mathcal{H}_q$. $\hat{x}_i$ acts on $\mathcal{H}_F$ through the diagonal representation π:

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}; \quad \pi(a^R) = \begin{pmatrix} a^R & 0 \\ 0 & a^R \end{pmatrix}; \quad a \in \mathcal{A}_F = \mathcal{H}_q \tag{6.42}$$

The completeness condition for the space $\mathcal{H}_F$ can be written more explicitly as,

$$\sum_{\mu=1}^{4} \left[\begin{pmatrix} \eta_\mu \\ 0 \end{pmatrix} (\eta_\mu^* \quad 0) + \begin{pmatrix} 0 \\ \eta_\mu \end{pmatrix} (0 \quad \eta_\mu^*) \right] = \mathbf{I}_{\mathcal{H}_F} \tag{6.43}$$

So we can write, in particular, for $\hat{x}_1$ as,

$$\pi(\hat{x}_1) = \pi(\hat{x}_1) I_{\mathcal{H}_F} = \sum_{\mu=1}^{4} \left[\begin{pmatrix} \hat{x}_1 \eta_\mu \\ 0 \end{pmatrix} (\eta_\mu^* \quad 0) + \begin{pmatrix} 0 \\ \hat{x}_1 \eta_\mu \end{pmatrix} (0 \quad \eta_\mu^*) \right] \tag{6.44}$$

A straightforward computation using (6.11, 6.36) and

$$\hat{x}_1 \eta_1 = \frac{\lambda}{2} \eta_3; \quad \hat{x}_1 \eta_2 = \frac{\lambda}{2} \eta_4; \quad \hat{x}_1 \eta_3 = \frac{\lambda}{2} \eta_1; \quad \hat{x}_1 \eta_4 = \frac{\lambda}{2} \eta_2 \tag{6.45}$$

yields

$$\begin{aligned} \pi(\hat{x}_1) = \frac{\lambda}{2} \Big[&|\phi_3)) ((\phi_1| + |\phi_4)) ((\phi_2| + |\phi_1)) ((\phi_3| + |\phi_2)) ((\phi_4| \\ &+ |\phi_7)) ((\phi_5| + |\phi_8)) ((\phi_6| + |\phi_5)) ((\phi_7| + |\phi_6)) ((\phi_8| \Big] \end{aligned} \tag{6.46}$$

This can be recast in the form of a matrix as

$$\pi(\hat{x}_1) = \frac{\lambda}{2} \begin{pmatrix} 0 & 0 & 1 & 0 & & & & \\ 0 & 0 & 0 & 1 & & & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & 1 & 0 & 0 & & & & \\ & & & & 0 & 0 & 1 & 0 \\ & & & & 0 & 0 & 0 & 1 \\ & & & & 1 & 0 & 0 & 0 \\ & & & & 0 & 1 & 0 & 0 \end{pmatrix} = \frac{\lambda}{2} \begin{pmatrix} \sigma_1 \otimes \mathbf{I}_2 & 0 \\ 0 & \sigma_1 \otimes \mathbf{I}_2 \end{pmatrix} = \frac{\lambda}{2} \mathbf{I}_2 \otimes (\sigma_1 \otimes \mathbf{I}_2) \tag{6.47}$$

where the rows and columns are labelled by $\phi_1, \phi_2, ..., \phi_8$ in this order.
Proceeding like-wise we get similar structure for $\pi(\hat{x}_2)$ and $\pi(\hat{x}_3)$, and enables us to write all these more compactly as

$$\pi(\hat{\vec{x}}) = \frac{\lambda}{2} \mathbf{I}_2 \otimes (\vec{\sigma} \otimes \mathbf{I}_2) \tag{6.48}$$

Coming to the right action $\pi(\hat{\vec{x}}^R)$, we again consider the first component $\pi(\hat{x}_1^R)$, which can be written like (6.44) as,

$$\pi(\hat{x}_1^R) = \pi(\hat{x}_1^R)\mathbf{I}_{\mathcal{H}_F} = \sum_{i=1}^{4} \begin{pmatrix} \hat{x}_1^R & 0 \\ 0 & \hat{x}_1^R \end{pmatrix} \left[\begin{pmatrix} \eta_i \\ 0 \end{pmatrix} (\eta_i^* \quad 0) + \begin{pmatrix} 0 \\ \eta_i \end{pmatrix} (0 \quad \eta_i) \right]$$

$$= \sum_{i=1}^{4} \left[\begin{pmatrix} \eta_i \hat{x}_1 \\ 0 \end{pmatrix} (\eta_i^* \quad 0) + \begin{pmatrix} 0 \\ \eta_i \hat{x}_1 \end{pmatrix} (0 \quad \eta_i) \right] \tag{6.49}$$

Now making use of the identities

$$\eta_1 \hat{x}_1 = \frac{\lambda}{2}\eta_2; \qquad \eta_2 \hat{x}_1 = \frac{\lambda}{2}\eta_1; \quad \eta_3 \hat{x}_1 = \frac{\lambda}{2}\eta_4; \quad \eta_4 \hat{x}_1 = \frac{\lambda}{2}\eta_3, \tag{6.50}$$

we readily obtain

$$\pi(\hat{x}_1^R) = \frac{\lambda}{2}\Big[|\phi_2))\,((\phi_1| + |\phi_1))\,((\phi_2| + |\phi_4))\,((\phi_3| + |\phi_3))\,((\phi_4| \\ + |\phi_5))\,((\phi_6| + |\phi_6))\,((\phi_5| + |\phi_7))\,((\phi_8| + |\phi_8))\,((\phi_7|\Big] \tag{6.51}$$

Again its matrix form can virtually read off as,

$$\pi(\hat{x}_1^R) = \frac{\lambda}{2} \begin{pmatrix} 0 & 1 & 0 & 0 & & & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 1 & 0 & & & & \\ & & & & 0 & 1 & 0 & 0 \\ & & & & 1 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 1 \\ & & & & 0 & 0 & 1 & 0 \end{pmatrix} = \frac{\lambda}{2} \begin{pmatrix} \mathbf{I}_2 \otimes \sigma_1 & 0 \\ 0 & \mathbf{I}_2 \otimes \sigma_1 \end{pmatrix} = \frac{\lambda}{2} \mathbf{I}_2 \otimes (\mathbf{I}_2 \otimes \sigma_1) \tag{6.52}$$

Furthermore, by following a similar process, we arrive to a more compact version of the corresponding right action, which is represented as,

$$\pi(\hat{\vec{x}}^R) = \frac{\lambda}{2}\mathbf{1}_2 \otimes (\mathbf{1}_2 \otimes \vec{\sigma}) \tag{6.53}$$

Now from the very structures of $\pi(\hat{\vec{x}})$ (6.48) and $\pi(\hat{\vec{x}}^R)$ (6.53), it is clear that

$$[\pi(\hat{x}_i), \pi(\hat{x}_j^R)] = 0 \qquad \forall \ i,j \ \in \{1,2,3\} \tag{6.54}$$

and

$$[\pi(\hat{x}_i^R), \pi(\hat{x}_j^R)] = -i\lambda\epsilon_{ijk}\pi(\hat{x}_k^R) \tag{6.55}$$

6.4 Determination of real structure $\mathcal{J}_F$

It is evident from table-2.1 given in chapter-2, that the spectral triple, in our ongoing discussion, must have even KO dimension as the grading operator γ_F exists (6.15). Thus, it automatically follows from the table-2.1 that, the real structure $\mathcal{J}_F$, if it exists, must commute with the Dirac operator D_F (6.15) [63, 168]:

$$[D_F, \mathcal{J}_F] = 0 \tag{6.56}$$

As a result, this operator ought to connect states with identical mass eigenvalues. There are almost no ambiguities for $m = \pm 1$, therefore one may certainly conclude, that under the action of $\mathcal{J}_F$ the Dirac spinors transform as follows:

$$|\pm 1, \frac{1}{2})) \quad \longleftrightarrow \quad \pm|\pm 1, -\frac{1}{2})) \tag{6.57}$$

Further in the paper [168] the author has indicated that the KO dimension of a triple should be 0 or 4 if the index (I) of the Dirac operator defined below is non-zero:

$$I = dim(H_+) - dim(H_-) \neq 0 \tag{6.58}$$

where H_+ is a subspace of the total Hilbert space $\mathcal{H}_F$, which corresponds to the positive eigen-space of the chirality operator γ_F and H_- represents the negative chirality subspace. In our case, it follows immediately, by using (6.24) and (6.25) that $I = 2 - 6 = -4 \neq 0$. Therefore, our search can be limited to KO dim = 0 and 4.

To express the transformations more explicitly (note that we have suppressed the double ket notation employed in (6.21)), for KO dimension 0 and 4 respectively, we have

$$\mathcal{J}_F\psi_1 = \psi_2; \quad \mathcal{J}_F\psi_3 = \psi_4; \quad \mathcal{J}_F\psi_2 = \psi_1; \quad \mathcal{J}_F\psi_4 = \psi_3 \tag{6.59}$$

or

$$\mathcal{J}_F\psi_1 = \psi_2; \quad \mathcal{J}_F\psi_3 = \psi_4 \quad \mathcal{J}_F\psi_2 = -\psi_1; \quad \mathcal{J}_F\psi_4 = -\psi_3 \tag{6.60}$$

Considering these possibilities, it becomes evident from the structures of $\psi_1, \ldots, \psi_8$ (see (6.21)) that a simple interchange: $\left\langle \frac{1}{2} \right| \longleftrightarrow \left\langle -\frac{1}{2} \right|$ and $\left| \frac{1}{2} \right\rangle \longleftrightarrow \left| -\frac{1}{2} \right\rangle$, followed by an interchange of the upper and lower components of all eight spinors $\psi_1, \ldots, \psi_8$ in (6.21) achieves the transformation described in (6.59) and (6.60). This transformation extends even to the massless sector as,

$$\mathcal{J}_F\psi_5 = \psi_6; \quad \mathcal{J}_F\psi_7 = \psi_8; \quad \mathcal{J}_F\psi_6 = \psi_5; \quad \mathcal{J}_F\psi_8 = \psi_7 \tag{6.61}$$

or

$$\mathcal{J}_F\psi_5 = \psi_6; \quad \mathcal{J}_F\psi_7 = \psi_8; \quad \mathcal{J}_F\psi_6 = -\psi_5; \quad \mathcal{J}_F\psi_8 = -\psi_7 \tag{6.62}$$

The above interchange operation in $\mathcal{H}_c$ implies the following swapping operation at the level of $\mathcal{H}_q$:

$$\Big|\frac{1}{2}\Big\rangle\Big\langle\frac{1}{2}\Big| \longleftrightarrow \Big|-\frac{1}{2}\Big\rangle\Big\langle-\frac{1}{2}\Big|; \quad \Big|\frac{1}{2}\Big\rangle\Big\langle-\frac{1}{2}\Big| \longleftrightarrow \Big|-\frac{1}{2}\Big\rangle\Big\langle\frac{1}{2}\Big| \tag{6.63}$$

Finally at the level of the total Hilbert space $\mathcal{H}_F = \mathbb{C}^2 \otimes \mathcal{H}_q$, this implies the swapping within the following four pairs of canonical basis vectors (6.11) :

$$\phi_1 \longleftrightarrow \phi_8; \quad \phi_2 \longleftrightarrow \phi_7; \quad \phi_3 \longleftrightarrow \phi_6; \quad \phi_4 \longleftrightarrow \phi_5 \tag{6.64}$$

The signs may vary depending on the KO dimension, whether it is 0 or 4. It is important to note that in (6.63), in addition to the swapping, we also had to exchange the upper and lower components. If a real structure $\mathcal{J}_F$ exists, its significance would be in its the ability to establish a mapping between the elements of the opposite algebra $\mathcal{A}_F^o$ and those of $\mathcal{A}_F$. This mapping can be expressed as follows:

$$\pi(a^o) = \mathcal{J}_F \pi(a^*) \mathcal{J}_F^*; \qquad a \in \mathcal{A}_F = \mathcal{H}_q, \text{and} \quad a^o \ \in \mathcal{A}_F^o = \mathcal{H}_q^o \tag{6.65}$$

fulfilling

$$[\pi(a), \pi(b^o)] = 0, \qquad \forall a, b \in \mathcal{A}_F \quad \text{(zero order condition)} \tag{6.66}$$

and, preferably, also

$$[[D_F, \pi(a)], \pi(b^o)] = 0 \qquad \text{(first order condition)} \tag{6.67}$$

In order to establish this, it is expected that $\hat{a}^o = \hat{a}^R$ can eventually be identified, thereby considering the opposite algebra $\mathcal{A}_F^o$ as generated by $\hat{x}_i^R$.
However, it turns out that there are no consistent solutions for a real structure $\mathcal{J}_F$ that satisfy the appropriate properties for KO dimension = 0 and (6.65). Therefore, we turn our attention to the transformations described in (6.60) and (6.62) and attempt to construct a real structure operator for KO dimension = 4.
To begin with, note that the Hilbert space $\mathcal{H}_q$ can be split into two subspaces: $\mathcal{H}_q = \mathcal{H}_q^{L_3=0} \oplus \mathcal{H}_q^{L_3 \neq 0}$. The former consists of eigenstates with $L_3 = 0$ (i.e., the linear span of η_1 and η_4), while the latter consists of eigenstates with $L_3 = \pm 1$ (i.e., the linear span of η_2 and η_3). Equation (6.36) indicates that the action of $\mathcal{J}_F$ should involve swapping between the eigen spinors in their respective sectors, while also interchanging the components of the $\mathbb{C}^2$ space, which corresponds to the lower and upper components of the spinors. To explicitly demonstrate the transformation sector-wise, we proceed as follows:
(1)

$$L_3 = 0 \text{ Sector} : \begin{pmatrix} \eta_1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ -\eta_4 \end{pmatrix}; \ \begin{pmatrix} \eta_4 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ -\eta_1 \end{pmatrix}; \ \begin{pmatrix} 0 \\ \eta_1 \end{pmatrix} \rightarrow \begin{pmatrix} \eta_4 \\ 0 \end{pmatrix}; \ \begin{pmatrix} 0 \\ \eta_4 \end{pmatrix} \rightarrow \begin{pmatrix} \eta_1 \\ 0 \end{pmatrix} \tag{6.68}$$

The interchange of η_1 and η_4 implies the following conversion in $\mathcal{H}_q$ (we have used (6.38)):

$$\hat{x}_3 \rightarrow -\hat{x}_3 \tag{6.69}$$

(2)

$$L_3 = \pm 1 \text{ Sector} : \begin{pmatrix} \eta_3 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix}; \ \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix} \rightarrow \begin{pmatrix} -\eta_3 \\ 0 \end{pmatrix} \tag{6.70}$$

Here $\eta_2 \leftrightarrow \eta_3$ implies the following interchanges in $\mathcal{H}_q$:

$$\hat{x}_+ \ \leftrightarrow \ \hat{x}_- \tag{6.71}$$

At this point it is tempting to build a real structure operator by inserting a complete space inversion followed by a hermitian conjugation while keeping the aforementioned interchanges in mind.
Let's try to use the space inversion first utilizing,

$$\hat{\vec{x}} \rightarrow \hat{x}' \rightarrow P\hat{\vec{x}} = -\hat{\vec{x}}, \tag{6.72}$$

where the following transformation in the η_μ basis (6.36) the action of the parity operator P in $\mathcal{H}_q$ can be implemented unitarily as,

$$\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\end{pmatrix} \to \begin{pmatrix}\eta'_1\\ \eta'_2\\ \eta'_3\\ \eta'_4\end{pmatrix} = \begin{pmatrix}\frac{4}{\lambda^2}\hat{x}_1\hat{x}_1^R & 0 & 0 & 0\\ 0 & e^{i\pi\hat{L}_3} & 0 & 0\\ 0 & 0 & e^{i\pi\hat{L}_3} & 0\\ 0 & 0 & 0 & \frac{4}{\lambda^2}\hat{x}_1\hat{x}_1^R\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 1\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\end{pmatrix} \tag{6.73}$$

so that the following relations

$$\eta'_1 = P\eta_1 = \eta_4;\ \eta'_2 = P\eta_2 = -\eta_2;\ \eta'_3 = P\eta_3 = -\eta_3;\ \eta'_4 = P\eta_4 = \eta_1 \tag{6.74}$$

hold, where the transformation (6.72) becomes quite obvious if we use (6.74).
At this point, however, it is important to note that the initial commutator bracket (6.1), which only fulfills an SO(3) symmetry, is not invariant under the transformation (6.72). In simpler terms, if the structure constant ϵ_{ijk} is considered as a pseudo tensor and held invariant under parity transformation, the commutation relation (6.1) will not be invariant. However, if we elevate ϵ_{ijk} to E_{ijk}, which is now a tensor fulfilling the following condition,

$$\begin{aligned} E_{ijk} &= \epsilon_{ijk}, \quad \text{for right handed system}\\ &= -\epsilon_{ijk} \quad \text{for left handed system}\end{aligned}$$

we can also make it respect the extended O(3) symmetry, where the definition of $\epsilon_{123} = +1$ in an usual right-handed system is used. In light of this, we would point out that the unital algebra $\mathcal{A}_F$ (6.4) for our $n = \frac{1}{2}$ instance may be written as

$$\mathcal{A}_F = Span_{\mathbb{C}}\{\eta_1, \eta_2, \eta_3, \eta_4\} = Span_{\mathbb{C}}\{I_2, \hat{x}_1, \hat{x}_2, \hat{x}_3\} \tag{6.75}$$

follows the multiplication rule:

$$\hat{x}_i\hat{x}_j = \frac{\lambda^2}{4}\delta_{ij}I_2 + \frac{i\lambda}{2}\epsilon_{ijk}\hat{x}_k \quad \in \ \mathcal{A}_F \tag{6.76}$$

appropriate for a right handed system. Particularly for $i \neq j$, in a right handed system,

$$\hat{x}_1\hat{x}_2 = \frac{i\lambda}{2}\hat{x}_3, \quad \text{and cyclic permutation}$$

while for the left handed system

$$\hat{x}_1\hat{x}_2 = -\frac{i\lambda}{2}\hat{x}_3, \quad \text{and cyclic permutation} \tag{6.77}$$

It should be noted that any generic O(3) element fulfilling det=-1 can produce the left handed system from the right handed system. One such instance is the parity operator discussed above that was first proposed in (6.72).Therefore, the multiplication rule, defined as follows for a left-handed system,

$$\hat{x}_i\hat{x}_j = \frac{\lambda^2}{4}\delta_{ij}I_2 - \frac{i\lambda}{2}\epsilon_{ijk}\hat{x}_k \quad \in \ \mathcal{A}_F \tag{6.78}$$

may now be 'upgraded' to the following new multiplication rule that combines the properties of the right-handed (6.76) and left-handed (6.78) systems as,

$$\hat{x}_i\hat{x}_j = \frac{\lambda^2}{4}\delta_{ij}I_2 + \frac{i\lambda}{2}E_{ijk}\hat{x}_k \quad \in \quad \mathcal{A}_F \tag{6.79}$$

This ensures,

$$P(\hat{x}_i\hat{x}_j) = \hat{x}_i\hat{x}_j \tag{6.80}$$

The multiplication rule has now changed, while maintaining the same underlying vector space structure of the algebra. As a result, the coordinate algebra given by (6.1) remains invariant under the action of the space inversion operator P, and the total symmetry group is enhanced from SO(3) to O(3), as mentioned earlier.
It is worth noting that the relation given by (6.76) is more stringent than (6.1) in the sense that (6.1) can be derived from (6.76) by simple anti-symmetrization. However, the converse is not true; (6.76) does not hold for fuzzy S^2_* associated with any $n \geq 1$ representation. For instance, in the case of $n = 1$, one can choose $(\rho_1(\hat{x}_i))_{jk} = -i\lambda\epsilon_{ijk}$, which satisfies (6.1), but does not satisfy (6.76).
In fact, the O(3)-symmetric Lie algebra satisfied by $\hat{x}_i$'s can be obtained simply by anti-symmetrizing (6.79), resulting in,

$$[\hat{x}_i, \hat{x}_j] = i\lambda E_{ijk}\hat{x}_k \tag{6.81}$$

The corresponding Lie-algebra satisfied by the right action $\hat{x}_i^R$ is therefore given by,

$$[\hat{x}_i^R, \hat{x}_j^R] = -i\lambda E_{ijk}\hat{x}_k^R \tag{6.82}$$

The resulting algebra, which exhibits an enlarged O(3) symmetry, is satisfied by an O(3)-covariant fuzzy sphere. This aspect has been studied previously in [170], where it was demonstrated that an O(3)-covariant fuzzy sphere can be constructed by imposing an energy cutoff on a quantum particle confined within a potential of the form $V(r)$.

Hence, it becomes evident that this parity symmetry should also be implemented through an automorphism symmetry of the algebra $\mathcal{A}_F$. This can be achieved by requiring the elements of $\mathcal{A}_F$ to transform as scalars under parity transformation (6.72):

$$a(I_2, \hat{\vec{x}}) \to a'(I_2, \hat{\vec{x}}') = a(I_2, \hat{\vec{x}}); \quad \hat{\vec{x}}' := -\hat{\vec{x}} \tag{6.83}$$

Next, we can implement the transformation (6.71) in the Hilbert space $\mathcal{H}_q$ using a Hermitian conjugation operator H (also referred to as the involution operator) or the $*$ operation introduced below (6.9). Furthermore, we require an operator that can act on the $\mathbb{C}^2$ sector of the total Hilbert space, interchanging the upper and lower components.
With all the necessary ingredients at our disposal, we can now introduce the real structure $\mathcal{J}_F$ as follows:

$$\mathcal{J}_F = (i\sigma_2) \otimes (H \circ P) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes (H \circ P) \tag{6.84}$$

where P (6.72,6.73) acts linearly. It can be verified that $\mathcal{J}_F$ is an anti-unitary and anti-linear operator.

To illustrate its action, we consider the example of $\mathcal{J}_F\psi_1 = -\psi_2$, where ψ_1 and ψ_2 are eigen spinors (as given in 6.21) of the Dirac operator $\mathcal{D}_F$. In order to facilitate this demonstration, we find it convenient to express ψ_1 and ψ_2 in terms of the basis η_μ, resulting in,

$$\psi_1 = N\begin{pmatrix}\eta_1 + (\sqrt{3}-1)\eta_4 \\ (2-\sqrt{3})\eta_2\end{pmatrix} \text{ and } \psi_2 = N\begin{pmatrix}(2-\sqrt{3})\eta_3 \\ \eta_4 + (\sqrt{3}-1)\eta_1\end{pmatrix}; \quad N = \frac{1}{3-\sqrt{3}} \tag{6.85}$$

Then

$$\begin{aligned}\mathcal{J}_F\psi_1 &= N\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}(H\circ P)(\eta_1 + (\sqrt{3}-1)\eta_4) \\ (H\circ P)(2-\sqrt{3})\eta_2\end{pmatrix} = N\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\eta_4 + (\sqrt{3}-1)\eta_1 \\ -(2-\sqrt{3})\eta_3\end{pmatrix}\\ &= -N\begin{pmatrix}(2-\sqrt{3})\eta_3 \\ \eta_4 + (\sqrt{3}-1)\eta_1\end{pmatrix} = -\psi_2\end{aligned} \tag{6.86}$$

where we have made use of (6.32) and the following set of relations:

$$(H\circ P)\eta_1 = \eta_4; \ \ (H\circ P)\eta_2 = -\eta_3; \ \ (H\circ P)\eta_3 = -\eta_2; \ \ (H\circ P)\eta_4 = \eta_1 \tag{6.87}$$

Proceeding in a similar manner, we get

$$\mathcal{J}_F\psi_2 = \psi_1;\ \mathcal{J}_F\psi_3 = -\psi_4;\ \mathcal{J}_F\psi_4 = \psi_3;\ \mathcal{J}_F\psi_5 = \psi_6;\ \mathcal{J}_F\psi_6 = -\psi_5;\ \mathcal{J}_F\psi_7 = -\psi_8;\ \mathcal{J}_F\psi_8 = \psi_7; \tag{6.88}$$

This implies

$$\mathcal{J}_F^2 = -1 \tag{6.89}$$

consistent with KO dimension-4 (See table- 2.1). We now need to check the property (6.65) with our real structure operator (6.84). For that purpose, let us take the example of $(H\circ P)(\hat{x}_i\eta_1)$. Using (6.76, 6.80), this may be recast simply as the following,

$$(H\circ P)(\hat{x}_i\eta_1) = (H\circ P)(\frac{\hat{x}_i}{2} + \frac{\hat{x}_i\hat{x}_3}{\lambda}) = H(-\frac{\hat{x}_i}{2} + \frac{\hat{x}_i\hat{x}_3}{\lambda}) = -\frac{\hat{x}_i}{2} + \frac{\hat{x}_3\hat{x}_i}{\lambda} = -\hat{x}_i^R\eta_4 \tag{6.90}$$

Similarly we get for other components of η_μ the following relations,

$$(H\circ P)(\hat{x}_i\eta_2) = \hat{x}_i^R\eta_3;\ \ (H\circ P)(\hat{x}_i\eta_3) = \hat{x}_i^R\eta_2;\ \ (H\circ P)(\hat{x}_i\eta_4) = -\hat{x}_i^R\eta_1 \tag{6.91}$$

These relations help us to obtain the actions of $\mathcal{J}_F$ on the canonical basis of $\mathcal{H}_F$ (6.11), which can now be seen to be given by the following relations using (6.87) :

$$\begin{aligned}&\mathcal{J}_F|\phi_1)) = -|\phi_8));\ \ \mathcal{J}_F|\phi_2)) = |\phi_7));\ \ \mathcal{J}_F|\phi_3)) = |\phi_6));\ \ \mathcal{J}_F|\phi_4)) = -|\phi_5))\\ &\mathcal{J}_F|\phi_5)) = |\phi_4));\ \ \mathcal{J}_F|\phi_6)) = -|\phi_3));\ \ \mathcal{J}_F|\phi_7)) = -|\phi_2));\ \ \mathcal{J}_F|\phi_8)) = |\phi_1))\end{aligned} \tag{6.92}$$

Now one can write down using (6.44),

$$\mathcal{J}_F\pi(\hat{x}_1)^\dagger\mathcal{J}_F^\dagger = \mathcal{J}_F\pi(\hat{x}_1)\mathbf{1}_{\mathcal{H}_\mathcal{F}}\mathcal{J}_F^\dagger = \mathcal{J}_F\sum_{i=1}^{8}\pi(\hat{x}_1)|\phi_i))\,((\phi_i|\,\mathcal{J}_F^\dagger \tag{6.93}$$

For the explicit illustration, let's use the first term from the right-hand side of the equation presented above as a sample,

$$\begin{aligned}
\mathcal{J}_F\pi(\hat{x}_1)|\phi_1))((\phi_1|\mathcal{J}_F^\dagger &= -\mathcal{J}_F\begin{pmatrix}\hat{x}_1\eta_1\\0\end{pmatrix}((\phi_8| \\
&= -\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\otimes(H\circ P)\begin{pmatrix}\hat{x}_1(\frac{1}{2}+\frac{\hat{x}_3}{\lambda})\\0\end{pmatrix}((\phi_8| \\
&= -\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\otimes H\begin{pmatrix}\frac{-\hat{x}_1}{2}+\frac{\hat{x}_1\hat{x}_3}{\lambda}\\0\end{pmatrix}((\phi_8| \\
&= -\begin{pmatrix}0\\\frac{\hat{x}_1}{2}-\frac{\hat{x}_3\hat{x}_1}{\lambda}\end{pmatrix}((\phi_8| = -\begin{pmatrix}0\\\eta_4\hat{x}_1\end{pmatrix}((\phi_8| = -|\phi_7))((\phi_8| \qquad (6.94)
\end{aligned}$$

Proceeding similarly for every term in the right hand side of (6.93), we get,

$$\begin{aligned}
\mathcal{J}_F\pi(\hat{x}_1)^\dagger\mathcal{J}_F^\dagger = -\frac{\lambda}{2}\,[|\phi_2))((\phi_1| + |\phi_1))((\phi_2| + |\phi_4))((\phi_3| + |\phi_3))((\phi_4| \\
+|\phi_5))((\phi_6| + |\phi_6))((\phi_5| + |\phi_7))((\phi_8| + |\phi_8))((\phi_7|] = -\pi(\hat{x}_1^R) \qquad (6.95)
\end{aligned}$$

For all the algebra generators $\hat{x}_i$ we can readily show

$$\pi(\hat{x}_i^o) := \mathcal{J}_F\pi(\hat{x}_i)^\dagger\mathcal{J}_F^\dagger = -\pi(\hat{x}_i^R) \qquad (6.96)$$

As a result, we may identify $\hat{x}_i^o = -\hat{x}_i^R$ rather than $\hat{x}_i^o = \hat{x}_i^R$, contrary yo what we had originally anticipated. But the negative sign turns out to be safe as it simply shows that elements of the opposite algebra $\mathcal{A}_F^o$ is also now subjected to the parity transformation as

$$\hat{x}_i^o = -\hat{x}_i^R = P\hat{x}_i^R \qquad (6.97)$$

As a result, both $\hat{\vec{x}}^o$ and $\hat{\vec{x}}^R$ act from the right and thus belong to $\mathcal{A}_F^o$. However, while $\hat{\vec{x}}^R$ satisfies the commutation algebra (6.82) appropriate for the right-handed system, $\hat{\vec{x}}^o$ will satisfy the commutation algebra of the left-handed system.
However, since the more general form of the commutation relation is now given by (6.82), which satisfies the entire O(3) symmetry, it immediately follows that $\hat{x}_i^o$ will also satisfy the same $\mathfrak{su}(2)^R$ Lie algebra (6.55).
It therefore follows trivially the following commutator relation is also satisfied.

$$[\mathcal{J}_F\pi(\hat{x}_i)^\dagger\mathcal{J}_F^\dagger, \mathcal{J}_F\pi(\hat{x}_j)^\dagger\mathcal{J}_F^\dagger] = [\pi(\hat{x}_i^R), \pi(\hat{x}_j^R)] = -i\lambda\epsilon_{ijk}\pi(\hat{x}_k^R) \qquad (6.98)$$

Finally, we would like to mention that our orbital angular momentum (6.19) changes sign under parity, which is consistent with the new definition of angular momentum in a commutative context ($\mathbb{R}^3$) as $L_i = E_{ijk}\hat{x}_j\hat{p}_k$, rather than $\epsilon_{ijk}\hat{x}_j\hat{p}_k$. This change in sign forces the occurrence of σ_i in the spin part to flip sign as well, resulting in the chirality operator (6.15) being an even object under parity. Similarly, if we replace ϵ_{ijk} with E_{ijk} in the Dirac operator (6.15), it too becomes even under parity.

6.4.1 Violation of the first-order condition

The requirement of the first-order condition, given by $[[D_F, \pi(a)], \pi(b^o)] = 0$, actually encodes the fact that the Dirac operator is a first-order differential operator and acts as a derivation of the algebra

$\mathcal{A}$ into itself, commuting with the entire opposite algebra $\mathcal{A}^o = \mathcal{J}\mathcal{A}\mathcal{J}^*$ i.e. belonging to its commutant. In [67], it was shown that by violating the first-order condition, the fluctuated Dirac operator $\mathcal{D}_A$ becomes non-invariant under inner fluctuations, and to restore its invariance, a quadratic inner fluctuation term needs to be added to $\mathcal{D}_A$.
Now, let us examine whether this condition is valid or not in our context. For that, let us choose, $\pi(a) = \pi(\hat{x}_l) = diag(\hat{x}_l, \hat{x}_l)$ and $\pi(b^o) = \pi(\hat{x}_p^o) = diag(\hat{x}_p^o, \hat{x}_p^o)$. Now using the form of the Dirac operator given in (6.15), we can show,

$$\begin{aligned}
[[D_F, \pi(\hat{x}_l)], \pi(\hat{x}_p^o)] &= \frac{i}{\lambda r_n}\gamma_F \Big[[\epsilon_{ijk}\sigma_i \otimes \hat{x_j}^R \hat{x}_k, \mathbf{I}\otimes \hat{x}_l], \mathbf{I}\otimes \hat{x}_p^R\Big] \\
&= -\frac{\gamma_F}{r_n}\Big[(\sigma_l \otimes \hat{x}_j^R \hat{x}_j - \sigma_i \otimes \hat{x}_l^R \hat{x}_i), \mathbf{I}\otimes \hat{x}_p^R\Big] \\
&= -\frac{i\lambda\gamma_F}{r_n}\Big(\epsilon_{lpm}\sigma_i \hat{x}_i \hat{x}_m^R - \epsilon_{jpm}\sigma_l \hat{x}_j \hat{x}_m^R\Big) \neq 0 \qquad (6.99)
\end{aligned}$$

So the first order condition is violated.

6.5 Chapter summary

In this chapter, following Watamura's structural forms [69] of Dirac and grading operators, we have provided a consistent formulation of an even and real spectral triple for the fuzzy sphere in its 1/2 representation. We provided precise forms for the eigen spinors of the SU(2) covariant forms of the chirality and Dirac operators. In addition, we showed how to get the real structural operator in the spin-1/2 representation, which is in agreement with the spectral data of KO dimension 4. For doing so, we had to enlarge the symmetry group of the fuzzy sphere from SO(3) to O(3). It should be observed that the first-order condition, a crucial component of the spectral formulation of the standard model, is violated by this specific spectral triple.

This opens up possibilities to explore phenomena beyond the scope of the usual standard model. Using the almost commutative geometric framework and this formulation, we can try to construct an SU(2) gauge theory. For that, we intend to study a model, where the fuzzy sphere represent the internal space in a toy model that combines a 4D commutative manifold with an algebra of the type $C^\infty(M, M_2(\mathbb{C}))$. Additionally, the fluctuated Dirac operator could contribute to the definition of the "Higgs field", opening up opportunities for building plausible cosmological models. For instance, one may investigate the interaction of gravity with dark matter candidates at higher energies [171].

Last but not least, it would be intriguing to contrast the effective action resulting from the spectral action principle with that of the conventional Kaluza-Klein theory, where the spacetime manifold is assumed to be the product of $M_4 \times S^2$ [172], with S^2 being a commutative 2-sphere. Such a comparison should throw light on their divergent characteristics and ramifications.

Chapter 7
Conclusions

The primary aim of this thesis has been to explore theories of NCG that offer intriguing novel features at the quantum level, while also establishing connections between NC theories and their commutative counterparts. In this final chapter we shall summarize briefly our findings and possible future directions arising from our results.

In chapter-1, after discussing the basic tenets of NCG and its major applications across different branches of physics associated to hugely different energy scales like QG corresponding to Planck scale physics to all the way to low energetic condensed matter phenomena like Quantum Hall effect to the "intermediate" level with associated energy scale ~ 10 TeV relevant for the existing SM of particle physics, we have briefly reviewed the Hilbert-Schmidt operatorial formulation and spectral triple formulation of NCG in chapter-2.

In chapter-3, we have studied effect of NC space-time in a non-relativistic time-dependent quantum system (forced harmonic oscillator) and found that noncommutativity can give rise to non-trivial geometric phase shift. In [112], the authors had shown an interesting relationship between the Fubini-Study metric defined in the projective Hilbert space of a time dependent quantum system and the energy-time uncertainty relation. Later, authors in [113] have pointed out that, the coherent states provide a natural framework for the realization of the above mentioned relationship and demonstrated this explicitly in their computation involving a spin system in a magnetic field and a generalized harmonic oscillator system respectively. In light of our work, it seems that we can also extend our study to compute the energy-time uncertainty relation in our system as well, by making use of the existing Fubini-Study metric of the projective Hilbert space. Presumably, the space-time noncommutativity, which is solely responsible for giving rise to the geometric phase and inducing a geometry in the parameter space, should have some tangible effect in the energy-time uncertainty relation and it should be quite interesting to investigate this point in our future work.

The formulation of QM in NC space-time necessitates the formulation of QFT within the framework of quantum space-time. In this regard, it becomes crucial to embrace the operatorial nature of space-time coordinates, rather than relegating them to mere classical numbers and employing star products. This allows us to circumvent the ambiguities arising from the non-uniqueness of star products as discussed in chapter-2. To achieve this, the Hilbert-Schmidt operatorial formulation of quantum mechanics proves indispensable at least for Moyal type of spaces or space-times. My future interest is to use this operatorial formalism to build up second quantization procedure in NC space-time.

In chapter-4, we have discussed the deformed Poincare symmetry of a NC space-time with Lie algebraic noncommutativity, namely the κ-Minkowski space-time and found the Lagrangian of a relativistic spinless massive particle in such background in a purely classical analysis. Subsequently,

we have uncovered a relation between the momentum dependent phase space algebra (which can be given the status of a generalized uncertainty principle or GUP) arising as a result of Lie algebraic space-time non-commutativity. The enveloping algebra-valued nature of phase-space algebra is then corroborated independently by Heisenberg-double construction in a Hopf algebroid framework. This can then be interpreted as the manifestation of non-trivial geometry of the corresponding momentum space and can be taken to be an effective description in a regime of QG, where there is no length scale and the noncommutativity arises purely from the inverse of a mass scale. Unlike [60], where the 4D curved momentum space represented by AN(3) group manifold which is a part of 4D de-Sitter space and the free particle Lagrangian is obtained by using group theoretical approach, appropriate for the AN(3) group manifold, we have employed the deformed actions of the undeformed Poincare symmetry of the κ-Minkowski manifold to write down the free particle Lagrangian so that the dynamical model remains invariant under the deformed transformation of the space-time coordinates. Contrary to [60], where the translation generators are not 4-vectors, in our model the translation generator P^μ's transform as 4-vectors, which is the novelty of our construction. This, in turn, helps us to label single particle states by mass and spin just like the commutative case. We then derived the deformed dispersion relation for this single particle system from the deformed geodesic distance in the curved momentum space. The deformed mass-shell condition enables us to identify a renormalized mass M moving in a κ-Minkowski space-time, which is related to its usual 'bare' mass m. And for this, we consider three cases, i.e. for time-like, light-like and space-like deformation parameter $\mathfrak{a}^\mu$. In the case of a light-like deformation parameter, no deformation is observed in the bare mass: $M = m$. However, when considering a space-like deformation parameter, the bare mass is found to have an upper bound of $m < \frac{1}{\mathfrak{a}}$. Correspondingly, the renormalized mass M is also found to be bounded above as, $M < \frac{\pi}{2\mathfrak{a}}$. On the other hand, for a time-like deformation parameter, M becomes a monotonically increasing function of m. Although there is no explicit upper bound on the masses m or M in this scenario, the existing mass scale serves as a critical point, which is also expected to be on the order of the Planck mass. Beyond this scale, the growth of M becomes nearly independent of that of m, resulting in the renormalized mass reaching effectively a plateau within a narrow range.
These observations raise captivating questions about the nature of spacetime itself and the measurements conducted within the domain of quantum gravity. Our work opens up the door to investigate the nature of many particle interaction in light of the curved momentum space and look for any signature, which can perhaps be attributed to "relative locality" [52]. It would also be interesting to take up the case of a relativistic spinning particle in κ-Minkowski background.

In chapter-5 and 6, we explored the spectral triple aspect of NCG and their possible application in beyond "almost commutative" geometrical framework for Standard Model of particle physics.
In chapter-5, after briefly discussing about the spectral distance of Euclidean Moyal plane, we have discussed the Lorentzian formulation of spectral triple *a la'* N. Franco [153, 154]. We then make use of the formulation to derive the spectral distance between two pure states separated by time-like intervals. This should pave the way for realistic model building where almost-commutative spaces are upgraded to fully non-commutative spaces, so as to provide us a glimpse beyond standard model physics.

The almost commutative manifold *a la'* Connes [62] which is given by a product of a continuous four dimensional manifold and a finite space/ non-commutative algebra namely $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ predicts, almost uniquely, the existing SM with all its fermions, gauge fields, Higgs field and their representations. A strong restriction on the non-commutative space results from the first order condition which came from the requirement that the Dirac operator is a differential operator of order one [45]. However, when one relaxes this condition, that opens up a new possibilities to study phenomena beyond standard model as was discussed in [67, 68].

In chapter-6, we could introduce, for the first time, a real structure for a spin-1/2 fuzzy sphere so that it becomes *real* and even spectral triple. However, the above mentioned spectral triple violates the first order condition. In this context, Connes et.al. [67] has come up with necessary mathematical formalism for spectral triples where although the first order condition gets violated, it nevertheless eventually facilitates the development of a phenomenologically viable model like Pati-Salam model [68], which goes beyond standard model. This gave us a strong motivation to formulate a toy model of SU(2) gauge theory by taking the spin-1/2 fuzzy sphere as the finite space along with 4D commutative manifold so that the total algebra becomes $\mathcal{A} = C^{\infty}(M, M_2(\mathbb{C}))$. Now one can try to construct a bosonic spectral action using almost commutative geometry framework *a la'* Connes, which respects SU(2) gauge symmetry, to capture some of the aspects of physics beyond the standard model.
And for that we need to augment linearly fluctuated Dirac operator $D + A \pm JAJ^{-1}$ with gauge potential $A = \sum_i a_i[D, b_i] (a_i, b_i \in \mathcal{A})$ with appropriate non-linear corrections-the so called 'fluctuation of a fluctuation'. That is how we hope to first determine the gauge and Higgs fields content and later we can evaluate the spectral action using heat kernel expansion method and then analyze the resultant model.

Appendices

Appendix A
On the effect of space-time noncommutativity on a time-independent system

Let us take, for example, a time-independent generic Hamiltonian placed in a NC space-time.

$$H = \frac{\hat{p}_x^2}{2m} + V(\hat{X}) \tag{A.1}$$

By representing it in coherent state basis (3.35) we get the following Schrödinger equation as,

$$i\partial_t \psi_{phy}(x,t) = (x,t|\hat{H}|\psi_{phy}) = \Big[\frac{p_x^2}{2m} + V(X_\theta)\Big]\psi_{phy}(x,t); \qquad \psi_{phy}(x,t) \in L^2_*(\mathbb{R}^1) \tag{A.2}$$

Now, using (3.60), we get

$$i\partial_t \psi_{phy}(x,t) = \Big[\frac{p_x^2}{2m} + SV(x)S^{-1}\Big]\psi_{phy}(x,t)$$

which, when re-written in terms of $\psi_c(x,t) = S^{-1}\psi_{phy}(x,t) \in L^2(\mathbb{R}^1)$ (3.62), readily yields the usual commutative Schrödinger equation:

$$i\partial_t \psi_c(x,t) = \Big[\frac{p_x^2}{2m} + V(x)\Big]\psi_c(x,t) \tag{A.3}$$

demonstrating how the spectrum and therefore the system's dynamics are unaltered while the commutative Schrödinger equation is successfully recovered. Here, the transformation merely affects the wave function of the system: $\psi_c = S^{-1}\psi_{phy}$.

Appendix B

Dirac's constraint analysis for the Lagrangian (3.24)

We will here perform a Dirac's constraint analysis on the first order form of the Lagrangian (3.24) and then compute the Dirac's bracket among the configuration space variables, where space-time and their corresponding canonical momenta are regarded as configuration space variables,

$$L_f^{\tau,\theta} = p_\mu \dot{x}^\mu - \sigma(\tau)(p_t + H) + \frac{\theta}{2}\epsilon^{\mu\nu}p_\mu \dot{p}_\nu, \quad \mu,\nu = 0,1 \tag{B.1}$$

$\sigma(\tau)$ is an arbitrary Lagrange multiplier enforcing the constraint $(p_t + H) \approx 0$. The canonical momenta corresponding to x^μ, p_μ and $\sigma(\tau)$ is given as follows :

$$\pi^x_\mu = \frac{\partial L^{\tau,\theta}}{\partial \dot{x}^\mu} = p_\mu; \qquad \pi^\mu_p = \frac{\partial L^{\tau,\theta}}{\partial \dot{p}_\mu} = -\frac{\theta}{2}\epsilon^{\mu\nu}p_\nu; \qquad \pi_\sigma = \frac{\partial L^{\tau,\theta}}{\partial \dot{\sigma}} = 0 \tag{B.2}$$

The aforementioned equations (B.2) are understood to be primary constraints of the theory since it is clear that the canonical momenta are not connected to the generalised velocities and are given by

$$\Phi_{1,\mu} = \pi^x_\mu - p_\mu \approx 0, \quad \Phi^\mu_2 = \pi^\mu_p + \frac{\theta}{2}\epsilon^{\mu\nu}p_\nu \approx 0, \quad \Phi_3 = \pi_\sigma \approx 0 \tag{B.3}$$

The Poission brackets between the canonical pairs are following,

$$\{x^\mu, \pi^x_\nu\} = \delta^\mu{}_\nu, \quad \{p_\mu, \pi^\nu_p\} = \delta_\mu{}^\nu, \quad \{\sigma, \pi_\sigma\} = 1 \tag{B.4}$$

We can derive the Poission brackets between the constraints (B.3), using (B.4) as

$$\begin{aligned} &\{\Phi_3, \Phi_3\} = \{\Phi_3, \Phi_{1,\mu}\} = \{\Phi_3, \Phi^\mu_2\} = 0 \\ &\{\Phi_{1,\mu}, \Phi_{1,\nu}\} = 0; \qquad \{\Phi_{1,\mu}, \Phi^\nu_2\} = -\delta_\mu{}^\nu \qquad \{\Phi^\mu_2, \Phi^\nu_2\} = \theta\epsilon^{\mu\nu} \end{aligned} \tag{B.5}$$

We can also write the canonical Hamiltonian as

$$H_c = \sigma(\tau)(p_t + H) \tag{B.6}$$

It is evident that with all other constraints, Φ_3 yields zero brackets. It is therefore a first class constraint. Due to the non-vanishing Poission bracket between $\Phi_{1,\mu}, \Phi_{2,\mu}$, the remaining constraints are categorized as second class constraints. As a result, the Dirac brackets allow us to strongly implement the second class constraints. Before continuing, however, it should be noted that the total Hamiltonian may be expressed by adding all constraints to the canonical Hamiltonian H_c as,

$$H_T = \sigma(\tau)(p_t + H) + \lambda_{1,\mu}\Phi_{1,\mu} + \lambda_{2,\mu}\Phi_2^{\mu} + \lambda_3\Phi_3 \tag{B.7}$$

where $\lambda_{1,\mu}, \lambda_{2,\mu}$ and λ_3 are suitable multipliers. The secondary constraint now results from the preservation of Φ_3 in time.

$$\Sigma = \dot{\Phi}_3 = \{H_T, \pi_\sigma\} = p_t + H \approx 0 \tag{B.8}$$

We can obtain the relationship between $\sigma(\tau)$ and $\lambda_{1,\mu}, \lambda_{2,\mu}$ from the time preservation criteria of the other constraints, which are unimportant in this case, . Instead, as was already said, we will implement the other second class constraints. Thus, in order to achieve that, we must first write down the constraint matrix Λ_{ab} and its inverse $(\Lambda^{-1})_{ab}$, which are now obtained as,

$$\Lambda_{ab} = \begin{pmatrix} \{\Phi_{1,\mu}, \Phi_{1,\nu}\} & \{\Phi_{1,\mu}, \Phi_2^{\nu}\} \\ \{\Phi_2^{\nu}, \Phi_{1,\mu}\} & \{\Phi_2^{\mu}, \Phi_2^{\nu}\} \end{pmatrix} = \begin{pmatrix} 0 & -\delta_\mu{}^{\nu} \\ \delta^{\nu}{}_{\mu} & \theta\epsilon^{\mu\nu} \end{pmatrix}; \quad (\Lambda^{-1})_{ab} = \begin{pmatrix} \theta\epsilon_{\mu\nu} & \delta^{\mu}{}_{\nu} \\ -\delta_\nu{}^{\mu} & 0 \end{pmatrix} \tag{B.9}$$

fulfilling, $\Lambda_{ab}(\Lambda^{-1})_{bc} = \delta_{ac}$.
Using the following definition for Dirac brackets [97, 98],

$$\{f, g\}_D = \{f, g\} - \{f, \Phi_a\}(\Lambda^{-1})_{ab}\{\Phi_b, g\} \tag{B.10}$$

we now compute the Dirac brackets between the phase space variables to get the following structure:

$$\{x^\mu, x^\nu\}_D = \theta\epsilon^{\mu\nu}, \quad \{p_\mu, p_\nu\}_D = 0, \quad \{x^\mu, p_\nu\}_D = \delta^{\mu}{}_{\nu} \tag{B.11}$$

These Dirac brackets easily produce the NCHA (3.26,3.27), which contains the space time NC structures, when raised to the quantum level.
Finally, we would like to make certain observations regarding the functions of the primary first class constraint Φ_3 and the secondary constraint Σ. The primary first-class constraint Φ_3 (B.3) deals with the non-physical momentum π_σ associated with $\sigma(t)$ that corresponds to the Lagrange multiplier. So, we may disregard this constraint. On the other hand, we can quickly demonstrate that the secondary constraint Σ (B.8) is now first class constraint since it shares vanishing Dirac brackets with all other primary constraints when we employ the Dirac brackets . Finally, it can be demonstrated that Σ generates the τ evolution of the system in the form of the theory's gauge transformation.

Appendix C
On the mapping S^{-1}: $L^2_*(\mathbb{R}^1) \rightarrow L^2(\mathbb{R}^1)$

Here we would like to prove that, the map (3.62) i.e. $S^{-1} : L^2_*(\mathbb{R}^1) \rightarrow L^2(\mathbb{R}^1)$, with $S^{-1} = e^{-\frac{\theta}{4}(\partial_t^2+\partial_x^2)}e^{i\frac{\theta}{2}\partial_t\partial_x}$, although being a non-unitary operator, is an inner product preserving map i.e. the inner product between a pair of arbitrary states in $L^2_*(\mathbb{R}^1)$ space with respect to star multiplication, is equal to the inner product of the corresponding transformed states belonging to $L^2(\mathbb{R}^1)$ with respect to ordinary point-wise multiplication.
For that, let $\psi_{phy}(x,t;\theta) \in L^2_*(\mathbb{R}^1)$ and $\psi_c(x,t) \in L^2(\mathbb{R}^1)$ related by (3.62) as $\psi_{phy}(x,t,\theta) = S\psi_c(x,t)$, satisfy the respective Schrödinger equations (3.59) and (3.67). Now, if we consider a stationary wave-function in $L^2(\mathbb{R}^1)$ as

$$\psi_c(x,t) = e^{-iEt}\psi_c(x), \tag{C.1}$$

then the corresponding physical state in $L^2_\star(R^1)$ can be easily determined by using (3.62) as

$$\psi_{phy}(x,t;\theta) = e^{-iEt}e^{\frac{\theta}{4}(-E^2+\partial_x^2)}e^{\frac{\theta}{2}E\partial_x}\psi_c(x) = e^{-iEt}\psi_{phy}(x;\theta) \tag{C.2}$$

So here too we see that, the time part gets factored out here in the form of e^{-iEt} just like (C.1), which is indicative of the fact that this $\psi_{phy}(x,t,\theta)$ is also a stationary state with the same energy eigenvalue E. Now the norm of the state $\psi_c(x,t) \in L^2(\mathbb{R}^1)$ is defined as

$$\begin{aligned}
&\int dx\ \psi_c^*(x,t)\psi_c(x,t) \\
&= \int dx\ (S^{-1}\psi_{phy}(x,t;\theta))^* \quad (S^{-1}\psi_{phy}(x,t;\theta)) \\
&= \int dx\ \left[\psi^*_{phy}(x,t;\theta)e^{-\frac{\theta}{4}(-E^2+\overleftarrow{\partial_x}^2)}e^{-\frac{i\theta}{2}\overleftarrow{\partial_t}\overleftarrow{\partial_x}}\right]\left[e^{-\frac{\theta}{4}(-E^2+\overrightarrow{\partial_x}^2)}e^{i\frac{\theta}{2}\overrightarrow{\partial_t}\overrightarrow{\partial_x}}\psi_{phy}(x,t;\theta)\right] \\
&= \int dx\ \left[\psi^*_{phy}(x,t;\theta)e^{-\frac{\theta}{4}(-\overleftarrow{\partial}_t\overrightarrow{\partial}_t-\overleftarrow{\partial_x}\overrightarrow{\partial}_x)}e^{\frac{i\theta}{2}\overleftarrow{\partial_t}\overrightarrow{\partial_x}}\right]\left[e^{-\frac{\theta}{4}(-\overleftarrow{\partial}_t\overrightarrow{\partial}_t-\overleftarrow{\partial_x}\overrightarrow{\partial_x})}e^{-i\frac{\theta}{2}\overleftarrow{\partial_x}\overrightarrow{\partial_t}}\psi_{phy}(x,t;\theta)\right] \\
&\quad \text{(Here we have used integration by parts and dropped the boundary terms.)} \\
&= \int dx\ \left[\psi^*_{phy}(x,t;\theta)e^{\frac{\theta}{2}(\overleftarrow{\partial}_t\overrightarrow{\partial}_t+\overleftarrow{\partial}_x\overrightarrow{\partial}_x)}e^{\frac{i\theta}{2}(\overleftarrow{\partial}_t\overrightarrow{\partial}_x-\overleftarrow{\partial}_x\overrightarrow{\partial}_t)}\psi_{phy}(x,t;\theta)\right] = \int dx\ \psi^*_{phy}(x,t;\theta)\star_V\psi_{phy}(x,t;\theta)
\end{aligned} \tag{C.3}$$

This allows us to demonstrate that the norm of the stationary wave function ψ_{phy} in $L^2_*(\mathbb{R}^1)$ space, associated with the energy eigenvalue E, is equal to the norm of the transformed stationary wave function ψ_c in $L^2(\mathbb{R}^1)$ space, associated with the same energy eigenvalue E. The typical point-wise multiplication and Voros star multiplication, respectively, are the products defining the norms in Hilbert spaces, $L^2(\mathbb{R}^1)$ and $L^2_*(\mathbb{R}^1)$ respectively. Now, let us construct a pair of non-stationary physical states by taking linear combination of stationary state wave functions with suitable coefficients such as,

$$\Psi_{phy}(x,t;\theta) = \sum_n A_n \psi^{(n)}_{phy}(x,t;\theta), \tag{C.4}$$

$$\Phi_{phy}(x,t;\theta) = \sum_n B_n \psi^{(n)}_{phy}(x,t;\theta) \tag{C.5}$$

where $\psi^{(n)}_{phy}(x,t;\theta) = e^{-iE_n t}\psi^{(n)}_{phy}(x;\theta)$ is a stationary state for a certain energy eigenvalue E_n of the form $\psi^{(n)}_{phy}(x;\theta)$. Because all bound stationary states in 1D space are non-degenerate, and as $\psi^{(n)}_{phy}(x,t;\theta)$ constitutes a complete set of basis, it can be chosen to be orthonormal as,

$$\int_t dx\ \psi^{*(n)}_{phy}(x,t;\theta) \star_V \psi^{(m)}_{phy}(x,t;\theta) = \delta_{mn} \tag{C.6}$$

Now let us consider the inner products between this pair of non stationary states (C.4) and (C.5) as

$$\int dx\ \Phi^{\star}_{phy}(x,t;\theta) \star_V \Psi_{phy}(x,t;\theta) = \sum_{n,m} A_m B^{\star}_n \int dx\ \psi^{(n)\star}_{phy}(x,t;\theta) \star_V \psi^{(m)}_{phy}(x,t;\theta) \tag{C.7}$$

Using the orthonormality (C.6) we arrive at

$$\begin{aligned} \int dx\, \Phi^{\star}_{phy}(x,t;\theta) \star_V \Psi_{phy}(x,t;\theta) &= \sum_n A_n B^{\star}_n \int dx\ \psi^{(n)\star}_{phy}(x,t;\theta) \star_V \psi^{(n)}_{phy}(x,t;\theta) \\ \text{(Now using (C.3))} \qquad &= \sum_n A_n B^{*}_n \int\ dx\ \psi^{(n)*}_c(x,t)\, \psi^{(n)}_c(x,t), \end{aligned} \tag{C.8}$$

where we have used $\psi^{(n)}_c(x,t) = S^{-1}\psi^{(n)}_{phy}(x,t;\theta)$.
It can be shown by simple argument that, if the non-stationary state $\Psi_{phy}(x,t,\theta) \in L^2_*(\mathbb{R}^1)$ is given by (C.4), then the corresponding non-stationary state in $L^2(\mathbb{R}^1)$ is given by,

$$\Psi_c(x,t) = S^{-1}\Psi_{phy}(x,t,\theta) = S^{-1}\sum_n A_n\psi^{(n)}_{phy}(x,t,\theta) = \sum_n A_n S^{-1}\psi^{(n)}_{phy}(x,t,\theta) = \sum_n A_n \psi^{(n)}_c(x,t) \tag{C.9}$$

Using this we can finally write (C.8) as,

$$\int\ dx\ \Phi^{*}_{phy}(x,t,\theta)\ \star_V\ \Psi_{phy}(x,t,\theta) = \int\ dx\ \Phi^{*}_c(x,t)\,.\,\Psi_c(x,t) \tag{C.10}$$

According to this equality, the inner product between a pair of non-stationary states in $L^2_*(\mathbb{R}^1)$ space with respect to star multiplication and the inner product between the corresponding pair of non-stationary states in $L^2(\mathbb{R}^1)$ space with respect to the standard point-wise product that we use for standard commutative QM, are indeed equal. This is also anticipated for the simple reason that the map, S^{-1}, is a non-unitary transformation that naturally modifies the inner-product of the Hilbert space and replaces the Voros star product in the $L^2_*(\mathbb{R}^1)$ space with the regular point-wise product of the $L^2(\mathbb{R}^1)$ space.

Appendix D
Derivation of the coproducts of deformed $\mathfrak{iso}(1,3)$ generators

Here we briefly sketch the method of obtaining the co-product for the generators $\hat{M}_{\mu\nu}, \hat{P}_\mu$. The action of $\hat{P}_\mu$ on a product of operator $\hat{X}$ valued functions $\hat{f}(\hat{X}).\hat{g}(\hat{X})$ can be shown to be given by

$$\begin{aligned}\hat{P}_\mu \triangleright (\hat{f}(\hat{X})\hat{g}(\hat{X})) &= ([\hat{P}_\mu, \hat{f}(\hat{X})]) \triangleright \hat{g}(\hat{X}) + (\hat{f}(\hat{X})\hat{P}_\mu) \triangleright \hat{g}(\hat{X}) \\ &= ([\hat{P}_\mu, \hat{f}(\hat{X})]) \triangleright \hat{g}(\hat{X}) + m((1 \otimes \hat{P}_\mu) \triangleright (\hat{f}(\hat{X}) \otimes \hat{g}(\hat{X}))\end{aligned} \tag{D.1}$$

where the sign $\triangleright$ denotes action of $\hat{P}_\mu$, given as

$$\hat{P}_\mu \triangleright \hat{f}(\hat{X}) = [\hat{P}_\mu, \hat{f}(\hat{X})] \triangleright \mathbf{1} \qquad \text{and} \quad \hat{P}_\mu \triangleright \mathbf{1} = 0 \tag{D.2}$$

and m is the multiplication map such that $m(f \otimes g) = f.g$ and one can immediately identify that

$$m\Big(\Delta\hat{P}_\mu \triangleright (\hat{f}(\hat{X}) \otimes \hat{g}(\hat{X})\Big) = [\hat{P}_\mu, \hat{f}(\hat{X})] \triangleright \hat{g}(\hat{X}) + m((1 \otimes \hat{P}_\mu) \triangleright (\hat{f}(\hat{X}) \otimes \hat{g}(\hat{X})) \tag{D.3}$$

So the coproduct of $\hat{P}_\mu$ can be obtained from the commutator relation $[\hat{P}_\mu, \hat{f}(\hat{X})]$. Now first taking $\hat{f}(\hat{X}) = \hat{X}_\nu$ as a simple example and using the relation (4.22) we obtain

$$\begin{aligned}[\hat{P}_\mu, \hat{X}_\nu] \triangleright \hat{g}(\hat{X}) &= [-i\eta_{\mu\nu}\phi + ia_\mu \hat{P}_\nu] \triangleright \hat{g}(\hat{X}) \\ &= [-i\eta_{\mu\nu}\phi + ia_\mu \phi^{-1}(\phi\eta_{\alpha\nu})\hat{P}^\alpha] \triangleright \hat{g}(\hat{X}) \\ &= (\hat{P}_\mu \triangleright \hat{X}_\nu)\phi \triangleright \hat{g}(\hat{X}) - a_\mu \Big[\phi^{-1}(\hat{P}_\alpha - \frac{a_\alpha}{2}F(\hat{P})), \hat{X}_\nu\Big] \triangleright \mathbf{1}(\hat{P}^\alpha \triangleright \hat{g}(\hat{X})) \\ &= (\hat{P}_\mu \triangleright \hat{X}_\nu)\phi \triangleright \hat{g}(\hat{X}) - a_\mu \Big[\phi^{-1}(\hat{P}_\alpha - \frac{a_\alpha}{2}F(\hat{P})) \triangleright \hat{X}_\nu\Big] \hat{P}^\alpha \triangleright \hat{g}(\hat{X})\end{aligned} \tag{D.4}$$

where we have used the following relations,

$$[\hat{P}_\mu, \hat{X}_\nu] \triangleright \mathbf{1} = -i\eta_{\mu\nu}, \qquad \Big[\hat{P}_\alpha - \frac{a_\alpha}{2}F(\hat{P}), \hat{X}_\nu\Big] = -i\eta_{\alpha\nu}\phi(\hat{P}) \tag{D.5}$$

where $F(\hat{P}) = \frac{2}{a^2}(1 - \sqrt{1 + a^2\hat{P}^2})$. We can now generalize the above relations to an arbitrary function $\hat{f}(\hat{X})$ using the method of induction on the monomials of $\hat{X}_\mu$ [122, 130] and write

$$[\hat{P}_\mu, \hat{f}(\hat{X})] = (\hat{P}_\mu \hat{f})\phi - a_\mu \Big(\hat{P}_\nu \phi^{-1}\hat{f}\Big)\hat{P}^\nu + \frac{a_\mu}{2}\Big(F(\hat{P})\phi^{-1}\hat{f}\Big)(a.\hat{P}) \tag{D.6}$$

So one can finally write the coproduct of $\hat{P}_\mu$ as

$$\Delta(\hat{P}_\mu) = \hat{P}_\mu \otimes \phi + \mathbf{1} \otimes \hat{P}_\mu - a_\mu(\hat{P}_\nu \phi^{-1}) \otimes \hat{P}^\nu + \frac{a_\mu}{2}(F(\hat{P})\phi^{-1}) \otimes (a.\hat{P})$$

Now one can also derive the coproduct of $\hat{M}_{\mu\nu}$ simply by using its realization given in (4.86) as

$$\begin{aligned}
&\Delta \hat{M}_{\mu\nu} \triangleright (\hat{f} \otimes \hat{g}) = \Delta\Big((\hat{X}_\mu \hat{P}_\nu - \hat{X}_\nu \hat{P}_\mu)\phi^{-1}(\hat{P})\Big) \triangleright (\hat{f} \otimes \hat{g}) \\
&= (\hat{M}_{\mu\nu}\hat{f}) \otimes \hat{g} + \hat{f} \otimes (\hat{M}_{\mu\nu}\hat{g}) + \Big[a_\mu\Big(\hat{P}^\lambda - \frac{a^\lambda}{2}F(\hat{P})\Big)\phi^{-1}\Big]\hat{f} \otimes (\hat{M}_{\lambda\nu}\hat{g}) - \\
&\Big[a_\nu\Big(\hat{P}^\lambda - \frac{a^\lambda}{2}F(\hat{P})\Big)\phi^{-1}\Big]\hat{f} \otimes (\hat{M}_{\lambda\mu}\hat{g})
\end{aligned}$$

which gives us the coproduct formula for $\hat{M}_{\mu\nu}$ stated in (4.27).
Co-associativity: One of the most important property of a Hopf algebra H is that it should be co-associative i.e. the coproduct Δ should satisfy

$$(\Delta \otimes \mathbf{1})\Delta = (\mathbf{1} \otimes \Delta)\Delta : H \to H \otimes H \otimes H \tag{D.7}$$

For example we can check via a lengthy but straight forward calculation, the following identity:

$$\begin{aligned}
(\Delta \otimes \mathbf{1})\Delta \hat{P}_\mu &= \hat{P}_\mu \otimes \phi \otimes \phi + \mathbf{1} \otimes \hat{P}_\mu \otimes \phi - a_\mu \hat{P}_\nu \phi^{-1} \otimes P^\nu \otimes \phi + \frac{a_\mu}{2}F(\hat{P})\phi^{-1} \otimes a.\hat{P} \otimes \phi \\
&+ \mathbf{1} \otimes \mathbf{1} \otimes \hat{P}_\mu - a_\mu \hat{P}_\nu \phi^{-1} \otimes \mathbf{1} \otimes \hat{P}^\nu - a_\mu \phi^{-1} \otimes \hat{P}_\nu \phi^{-1} \otimes \hat{P}^\nu \\
+\frac{a_\mu}{a^2}&\Big[\phi^{-1} \otimes \phi^{-1} \otimes a.\hat{P} - \mathbf{1} \otimes \mathbf{1} \otimes a.\hat{P} + a.\hat{P}\phi^{-1} \otimes \mathbf{1} \otimes a.\hat{P} + \phi^{-1} \otimes a.\hat{P}\phi^{-1} \otimes a.\hat{P}\Big] \\
&= (\mathbf{1} \otimes \Delta)\Delta \hat{P}_\mu
\end{aligned} \tag{D.8}$$

To prove the above identity we have used $F(\hat{P}) = \frac{2}{a^2}(1 - \sqrt{1 + a^2\hat{P}^2}) = \frac{2}{a^2}(1 - \phi + a.\hat{P})$ and the homomorphism of the coproduct i.e. $\Delta(a.b) = \Delta(a).\Delta(b),\ \forall a, b \in H$. We have also used the coproduct $\Delta(\phi^{-1}) = \phi^{-1} \otimes \phi^{-1}$, so ϕ^{-1} can be regarded as a group-like element of the Hopf algebra as ϕ itself i.e. $\Delta(\phi) = \phi \otimes \phi$. The coproduct of $\hat{M}_{\mu\nu}$ can also be verified to be co-associative: $(\Delta \otimes \mathbf{1})\Delta \hat{M}_{\mu\nu} = (\mathbf{1} \otimes \Delta)\Delta \hat{M}_{\mu\nu}$. It is important to notice that the set of commutator relations in (4.1,4.3) and (4.21,4.22) closes on the universal enveloping algebra generated by the enlarged set $\{\hat{M}_{\mu\nu}, \hat{P}_\mu, \hat{X}_\mu\}$. We have already discussed the coalgebra sector corresponding to the Poincare generators in chapter-4, section-4.1.1. It should now be augmented by the co-algebra structure of $\hat{X}_\mu$ itself. We, however, contend ourselves just by quoting the expression of coproduct of $\hat{X}_\mu$, given as [130]

$$\Delta \hat{X}_\mu = \phi \otimes \hat{X}_\mu + \hat{X}_\mu \otimes \mathbf{1} - a_\mu\Big[\hat{P}_\alpha - \frac{a_\alpha}{2}F\Big] \otimes \hat{X}^\alpha \tag{D.9}$$

as this expression is not used in the construction of Heisenberg double (see sec-4.1.1) on the way to provide an alternative 'derivation' of deformed Heisenberg algebra (4.22), for reasons explained there. In fact we have to make use of the undeformed i.e. the primitive form of coproduct :

$$\Delta_0(\hat{X}_\mu) = \hat{X}_\mu \otimes\ \mathbf{1} + \mathbf{1} \otimes \hat{X}_\mu, \tag{D.10}$$

as only translational generators $\hat{P}_\mu \in \hat{\mathcal{J}}$ will be used in the Heisenberg double construction, $\mathcal{H} = \mathcal{U}(\hat{\mathcal{J}}) \# \mathcal{U}(\hat{\mathcal{M}})$, where the Lorentz generators $\hat{M}_{\mu\nu}$ are excluded.

Appendix E
Symplectic analysis of free particle Lagrangian in κ-Minkowski space-time

Let us first write a general first-order Lagrangian as,

$$L_f = a_i(\xi)\dot{\xi}^i - H_c \tag{E.1}$$

where ξ_i are the phase space variables, $i = 1, ..., 2N$ for a N dimensional coordinate space. Note that $a_i(\xi)$ can be considered as a sort of vector potential (connection) for an abelian gauge theory, since modification of a_i by a total derivative as

$$a_i(\xi) \to a_i(\xi) + \frac{\partial\theta}{\partial\xi^i} \tag{E.2}$$

does not affect dynamics, since the Lagrangian changes by a total time-derivative term. However, the structure of constraints will vary under the transformation (E.2) as can be seen directly from (E.1). Now for a generic $a_i(\xi)$, one can find out the Euler-Lagrange equation from (E.1) as

$$f_{ij}(\xi)\dot{\xi}^j = \frac{\partial H_c}{\partial\xi^i} \tag{E.3}$$

where $f_{ij} = \frac{\partial a_j(\xi)}{\partial\xi_i} - \frac{\partial a_i(\xi)}{\partial\xi_j}$ acts as gauge invariant two form (curvature) constructed out of gauge variant connection $a_i(\xi)$ and is called symplectic two form. It can be shown that $f_{ij}(\xi)$ is basically the constraint matrix, which remains invariant under the gauge transformation of $a_i(\xi)$.

Now we carry out the symplectic analysis *a la'* Fadeev Jackiew (FJ) for the system Lagrangian and show that it indeed produces the same symplectic brackets as that of (4.46), as expected. Symplectic approach is an alternative and sometimes quicker method (than Dirac's analysis) specially for first order system to obtain the phase space brackets from equations of motion. We can calculate the Euler

Lagrange equation of motion from the system Lagrangian (4.56) as

$$\phi^{-1}\dot{P}_\mu + \frac{\phi^{-1}\mathfrak{a}_\mu}{\phi - \mathfrak{a}.P}(P.\dot{P}) = 0 \tag{E.4}$$

$$\begin{aligned}\phi^{-1}\dot{X}_\mu + \frac{\phi^{-1}}{\phi - \mathfrak{a}.P}(\mathfrak{a}.\dot{X})P_\mu + \phi^{-2}[\mathfrak{a}_\mu(X.\dot{P}) - X_\mu(\mathfrak{a}.\dot{P})] \\ + \frac{\phi^{-2}}{\phi - \mathfrak{a}.P}[\mathfrak{a}_\mu(\mathfrak{a}.X)(P.\dot{P}) - \mathfrak{a}^2 X_\mu(P.\dot{P}) - (\mathfrak{a}.\dot{P})(\mathfrak{a}.X)P_\mu + \mathfrak{a}^2(X.\dot{P})P_\mu] = e\ \frac{\partial f(P^2)}{\partial P^\mu}\end{aligned} \tag{E.5}$$

From FJ analysis, it is known that these equations of motion can be recast in the following form

$$\Lambda_{\mu\nu,ab}\dot{\xi}_{b,\nu} = \frac{\partial H_c}{\partial \xi^{a,\mu}} \tag{E.6}$$

where $\Lambda_{ab,\mu\nu}$ is basically the constraint matrix. So we can read off the components of constraint matrix from (E.6) using (E.5), which exactly matches with (4.53). Now the symplectic bracket between two variables f and g is given by

$$\{f, g\}_{SB} = (\Lambda^{-1})^{\mu\nu}{}_{ab}\ \partial_{\mu,a} f\ \partial_{\nu,b}\, g \tag{E.7}$$

where $\partial_{\mu,a} = \frac{\partial}{\partial \xi^\mu_a}$ and Λ^{-1} is the inverse of the constraint matrix given in (4.52). So the symplectic brackets between phase space variables are given by

$$\{X_\mu, X_\nu\}_{S.B} = (\mathfrak{a}_\mu X_\nu - \mathfrak{a}_\nu X_\mu) = \theta_{\mu\nu}; \quad \{P_\mu, X_\nu\}_{S.B} = -\eta_{\mu\nu}\phi(P) + \mathfrak{a}_\mu P_\nu; \quad \{P_\mu, P_\nu\}_{S.B} = 0 \tag{E.8}$$

It can be seen that, the symplectic brackets are identical with the Dirac brackets and produce the classical version of the κ Minkowski algebra (4.1).

Appendix F

Bargmann-Fock coherent state basis and their properties:

Here, we present some of the working formulas and notations for the Bargmann-Fock coherent state basis. Unlike Glauber-Sudarshan coherent states, the Bargmann-Fock basis is unnormalized. We define it as follows:

$$|z\rangle_B = e^{z\hat{b}^\dagger}|0\rangle, \quad {}_B\langle\bar{z}| = \langle 0|e^{\bar{z}b}, \text{giving } {}_B\langle\bar{z}|w\rangle_B = e^{\bar{z}w} \tag{F.1}$$

and is related to the normalised Glauber-Sudarshan coherent states (2.12) as $|z\rangle = e^{-\frac{|z|^2}{2}}|z\rangle_B$. The use of the Bargmann-Fock basis offers the advantage that vectors belonging to $\mathcal{H}_c$ (2.4) (and the dual space $\widetilde{\mathcal{H}_c}$) can be represented by anti-holomorphic functions $\psi(\bar{z}) := \langle\bar{z}|\psi\rangle$ (and holomorphic functions $\psi^*(z) := \langle\psi|z\rangle$, respectively). For simplicity, we will omit the subscript B when referring to the Bargmann-Fock basis from now on. The completeness relation is given by

$$\int d\mu(z,\bar{z})\ |z\rangle\langle\bar{z}| = \mathbf{1}, \ \text{ where } d\mu(z,\bar{z}) = e^{-|z|^2}\frac{Re(dz)Im(dz)}{\pi} \tag{F.2}$$

The overlap of this basis with a Fock state in (2.4) is given by

$$\langle\bar{z}|n\rangle = \frac{\bar{z}^n}{\sqrt{n!}}; \ \ \langle n|z\rangle = \frac{z^n}{\sqrt{n!}} \tag{F.3}$$

Another pair of important identities are,

$$\int d\mu(z,\bar{z})\langle\psi|z\rangle\langle\bar{z}|w\rangle = \langle\psi|w\rangle = \psi^*(w), \ \int d\mu(z,\bar{z})\langle\bar{w}|z\rangle\langle\bar{z}|\psi\rangle = \langle\bar{w}|\psi\rangle = \psi(\bar{w}) \tag{F.4}$$

This allows us to identify $\langle\bar{z}|w\rangle = e^{\bar{z}w} = \delta^2(\bar{z},w)$ as a delta function in this space. Moreover, we can introduce the concept of functional derivatives in this context. To do so, let us consider a functional $F[\psi(\bar{z})]$ that maps the space of anti-holomorphic functions $\psi(\bar{z})$ to complex numbers: $F : \psi(\bar{z}) \to \mathbb{C}$. The functional derivative $\frac{\delta F[\psi(\bar{z})]}{\delta\psi(\bar{w})}$ is then defined by the condition:

$$\int \frac{\delta F[\psi(\bar{z})]}{\delta\psi(\bar{w})}\phi(\bar{w})d\mu(w) = \left.\frac{dF[\psi(\bar{z}) + \lambda\phi(\bar{z})]}{d\lambda}\right|_{\lambda=0} \tag{F.5}$$

where $\phi(\bar{z})$ represents a small perturbaton with a strength λ. One can then easily evaluate the following functional derivatives:

$$\frac{\delta\psi(\bar{z})}{\delta\psi(\bar{w})} = \delta^2(\bar{z}, w) \quad \text{and} \quad \frac{\delta\psi^*(z)}{\delta\psi^*(w)} = \delta^2(\bar{w}, z) \tag{F.6}$$

where the second one follows from the first one by assuming complex conjugation, or alternatively defines similar functional derivative by adding suitable functionals to the space.

Appendix G

On finite dimensional matrix solution of (5.6) and some related observations

In [42] it has been shown that there exist a finite dimensional solution of optimal algebra element $\hat{a}_s$ saturating the ball condition $\|[\hat{b}, \hat{a}]\|_{op} \leq \sqrt{\frac{\theta}{2}}$. For computation of distance between harmonic oscillator states ρ_{n+1} and ρ_n, the finite dimensional optimal algebra can be taken as

$$\hat{a}_s = \frac{\sqrt{\theta}}{2\sqrt{2(n+1)}}\left[|n+1\rangle\langle n+1| - |n\rangle\langle n|\right] \tag{G.1}$$

If we compute the operator $[\hat{b}, \hat{a}]^\dagger[\hat{b}, \hat{a}]$ with this finite dimensional form of $\hat{a}_s$, we shall get,

$$[\hat{b}, \hat{a}]^\dagger[\hat{b}, \hat{a}] = \frac{\theta}{8(n+1)}[4(n+1)|n+1\rangle\langle n+1| + (n+2)|n+2\rangle\langle n+2| + n|n\rangle\langle n|] \tag{G.2}$$

It is apparent from the equation's form that the right-hand side matrix is diagonal but not proportional to the unit matrix. To compute the operator norm, we need to consider the highest eigenvalue, which is $4(n+1)$. This yields $\|[\hat{b}, \hat{a}]\|_{op} = \sqrt{\frac{\theta}{2}}$.

Therefore, we have demonstrated that a finite-dimensional form of $\hat{a}_s$ can also satisfy the ball condition. However, it is important to note that functional differentiation only provides information about local extrema. To obtain the operator norm, we must also determine the maximal eigenvalue. In this discrete case, the operator norm is not translationally invariant as it depends on n. It is employed to compute the distance between successive harmonic oscillator states ρ_n and ρ_{n+1}. Consequently, this distance is also non-invariant under translation and is given by the expression

$$d(\rho_n, \rho_{n+1}) = \sqrt{\frac{\theta}{2(n+1)}} \tag{G.3}$$

See [42, 161] for details.

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